Discussion 1A

Note: Not all problems will be covered in discussion. We recommend finishing the discussion worksheet on your own time. It is additionally worth noting that minor changes in notation are not important. e.g., putting commas after quantifiers, as in Q3. You may use parentheses instead, or put a period instead of a comma.

1. **Writing in propositional logic**

   For each of the following sentences, translate the sentence into propositional logic using the notation introduced in class, and write its negation.

   (a) The square of a nonzero integer is positive.

   (b) There are no integer solutions to the equation $x^2 - y^2 = 10$.

   (c) There is one and only one real solution to the equation $x^3 + x + 1 = 0$.

   (d) For any two distinct real numbers, we can find a rational number in between them.

2. **Implication**

   Which of the following implications are true? Give a counterexample for each false assertion.

   (a) $\forall x, \forall y, P(x,y) \implies \forall y, \forall x, P(x,y)$.

   (b) $\exists x, \exists y, P(x,y) \implies \exists y, \exists x, P(x,y)$.

   (c) $\forall x, \exists y, P(x,y) \implies \exists y, \forall x, P(x,y)$.

   (d) $\exists x, \forall y, P(x,y) \implies \forall y, \exists x, P(x,y)$.

3. **Perfect Square**

   A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

4. **Irrationals**

   Prove that $2^{1/n}$ is not rational for any integer $n > 3$. [Hint : Fermat's Last Theorem states that for all integers $n > 2$, there do not exist three positive integers a, b, c that satisfy $a^n + b^n = c^n$.]

5. **Pigeonhole Principle**

   Prove that if you put $n+1$ apples into n boxes, any way you like, then at least one box must contain at least 2 apples. This is known as the *pigeonhole principle*.

6. **Numbers of Friends**

   Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party.