Discussion 1A

Note: Not all problems will be covered in discussion. We recommend finishing the discussion worksheet on your own time. It is additionally worth noting that minor changes in notation are not important. e.g., putting commas after quantifiers, as in Q3. You may use parentheses instead, or put a period instead of a comma.

1. **Writing in propositional logic**

   For each of the following sentences, translate the sentence into propositional logic using the notation introduced in class, and write its negation.

   (a) The square of a nonzero integer is positive.

   (b) There are no integer solutions to the equation $x^2 - y^2 = 10$.

   (c) There is one and only one real solution to the equation $x^3 + x + 1 = 0$.

   (d) For any two distinct real numbers, we can find a rational number in between them.

   Solution:

   (a) We can rephrase the sentence as "if n is a nonzero integer, then $n^2 > 0$", which can be written as

   $$\forall n \in Z, (n \neq 0) \rightarrow (n^2 > 0)$$

   or equivalently as

   $$\forall n \in Z, (n = 0) \vee (n^2 > 0).$$

   The latter is easier to negate, and its negation is given by

   $$\exists n \in Z, (n \neq 0) \wedge (n^2 \leq 0)$$

   (b) The sentence is

   $$\forall x, y \in Z, x^2 - y^2 \neq 10.$$

   The negation is

   $$\exists x, y \in Z, x^2 - y^2 = 10$$

   (c) Let $p(x) = x^3 + x + 1$. The sentence can be read "there is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x." Or,

   $$\exists x \in \mathbb{R}, (p(x) = 0) \wedge (\forall y in \mathbb{R}, (p(y) = 0) \implies (x = y)).$$

   Its negation is given by

   $$\forall x \in \mathbb{R}, (p(x) \neq 0) \vee (\exists y \in \mathbb{R}, (p(y) = 0) \wedge (x \neq y)).$$

(d) The sentence can be read "if x and y are distinct real numbers, then there is a rational number z between x and y." Or,

$$\forall x,y \in \mathbb{R}, (x \neq y) \implies (\exists z \in \mathbb{Q}, (x < z < y) \vee (y < z < x)).$$

Equivalently,

$$\forall x,y \in \mathbb{R}, (x = y) \vee (\exists z \in \mathbb{Q}, (x < z < y) \vee (y < z < x)).$$

The negation is

$$\exists x,y \in \mathbb{R}, (x \neq y) \wedge (\forall z \in \mathbb{Q}, ((z \leq x) \vee (z \geq y)) \wedge ((y \geq z) \vee (x \leq z))).$$

2. **Implication**

Which of the following implications are true? Give a counterexample for each false assertion.

(a) $\forall x, \forall y, P(x,y) \implies \forall y, \forall x, P(x,y)$.

Solution: True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.

(b) $\exists x, \exists y, P(x,y) \implies \exists y, \exists x, P(x,y)$.

Solution: True. There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.

(c) $\forall x, \exists y, P(x,y) \implies \exists y, \forall x, P(x,y)$.

Solution: False. Let $P(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $P(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedant is true and the consequence is false, those the entire implication statement is false.

(d) $\exists x, \forall y, P(x,y) \implies \forall y, \exists x, P(x,y)$.

Solution: True. The first statement says that there is an x, say x' where for every y, $P(x,y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y. Note that the two statements are not equivalent as the converse of this is statement 3, which is false.

3. **Perfect Square**

Solution: (Proof) A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

Solution: Let $n = m^2$ for some integer m. Since n is odd, m is also odd, i.e., of the form $m = 2l+1$ for some integer l. Then, $m^2 = 4l^2 + 4l + 1 = 4l(l+1) + 1$. Since one of l and $l+1$ must be even, $l(l+1)$ is of the form $2k$ and $n = m^2 = 8k+1$.

4. **Irrationals**

Solution: (Contradiction) Prove that $2^{1/n}$ is not rational for any integer $n > 3$. [Hint : Fermat's Last Theorem states that for all integers $n > 2$, there do not exist three positive integers a, b, c that satisfy $a^n + b^n = c^n$.]

Solution: If not, then there exists an integer $n > 3$ such that $2^{1/n} = \frac{p}{q}$ where p, q are positive integers. Thus, $2q^n = p^n$, and this implies,

$$q^n + q^n = p^n$$

, which is a contradiction to the Fermat's Last Theorem.

5. **Pigeonhole Principle**

Solution: (Problem solving) Prove that if you put $n+1$ apples into n boxes, any way you like, then at least one box must contain at least 2 apples. This is known as the *pigeonhole principle*.

Solution: Suppose this is not the case. Then all the boxes would contain at most 1 apple. Then the maximum number of apples we could have would be n, but this is a contradiction since we have $n+1$ apples.

6. **Numbers of Friends**

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party.

Solution: **Answer:** Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \ldots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.