CS 70 Discrete Mathematics and Probability Theory
Fall 2016 Seshia and Walrand

Discussion 1B

1. **Power Inequality**

   Use induction to prove that for all integers $n \geq 1$, $2^n + 3^n \leq 5^n$.

2. **Bit String**

   Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

   $$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

   where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.

3. **Series**

   Prove that, for any positive integer n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

4. **Fibonacci for Home**

   Recall, the Fibonacci numbers, defined recursively as

   $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$.

   Prove that every third Fibonacci number is even. For example, $F_3 = 2$ is even and $F_6 = 8$ is even.

5. **Convergence of Series**

Use induction to prove that for all integers $n \geq 1$,

$$\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2.$$

Hint: Strengthen the induction hypothesis to $\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2 - \frac{1}{\sqrt{n}}$.

6. **Elephant Mosquito Paradox**

Claim: The weight of an elephant equals the weight of a mosquito.

Proof: Let x be the weight of an elephant, and y that of a mosquito. Call the sum of the two weights 2v, so that

$$x + y = 2v$$

From this equation we can obtain two more.

$$x - 2v = -y, x = -y + 2v$$

Multiplying those together, we get

$$x^2 - 2vx = y^2 - 2vy$$

Add v^2 to both sides.

$$x^2 - 2vx + v^2 = y^2 - 2vy + v^2$$
$$(x - v)^2 = (y - v)^2$$

Taking square roots, we get

$$x - v = y - v$$

From this we conclude: $x = y$. That is, the elephant's weight (x) equals the mosquito's weight (y). Q.E.D. What is wrong here? You only need to find one wrong step, but identify all the wrong steps if you find more than one.