1. **Power Inequality**

   Use induction to prove that for all integers $n \geq 1$, $2^n + 3^n \leq 5^n$.

   Solution: We use induction on n. The base case $n = 1$ is true because $2 + 3 = 5$. Assume the inequality holds for some $n \geq 1$. For $n + 1$, we can write:

$$\begin{aligned}
2^{n+1} + 3^{n+1} &= 2 \cdot 2^n + 3 \cdot 3^n \\
&< 3 \cdot 2^n + 3 \cdot 3^n \\
&= 3(2^n + 3^n) \\
&< 3 \cdot 5^n (*) \\
&< 5 \cdot 5^n \\
&< 5^{n+1}
\end{aligned}$$

   where the inequality (*) follows from the induction hypothesis. This completes the induction.

2. **Bit String**

   Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

   where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.

   Solution: Prove by strong induction on n. Note that this is the first time students will have seen strong induction, so it is important that this problem be done in an interactive way that shows them how simple induction gets stuck.

   - *Base Case:* $n = 1$ can be written with 1×2^0.
   - *Inductive Hypothesis:* Assume that the statement is true for all $1 \leq k \leq n$.

- *Inductive Step:* If $n+1$ is divisible by 2, then it can use the representation of $(n+1)/2$.

$$(n+1)/2 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0$$
$$n+1 = 2 \cdot (n+1)/2 = c_k \cdot 2^{k+1} + c_{k-1} \cdot 2^k + \ldots + c_1 \cdot 2^2 + c_0 \cdot 2^1.$$

  Otherwise, n must be divisible by 2 and have $c_0 = 0$. We can obtain the representation of $n+1$ from n.

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + 0 \cdot 2^0$$
$$n+1 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + 1 \cdot 2^0$$

  Therefore, the statement is true.

3. **Series**

   Prove that, for any positive integer n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

   Solution:

   - Base case: when $n = 1$, $\sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2\cdot 1+1)}{6}$.
   - Inductive hypothesis: assume for $n = k \geq 1$ that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.
   - Inductive step:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the inductive hypothesis)} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.
\end{aligned}$$

   By the principle of induction, the claim is proved.

4. **Fibonacci for Home**

Recall, the Fibonacci numbers, defined recursively as

$F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$.

Prove that every third Fibonacci number is even. For example, $F_3 = 2$ is even and $F_6 = 8$ is even.

Solution:

First, we should prove that all the fibonacci numbers are integer by induction: $P(k)$ is "F_k is an integer." This follows from the fact that F_1 and F_2 are integer, and the induction step follows from $F_k = F_{k-1} + F_{k-2}$, the (strong) induction hypothesis that F_{k-1} and F_{k-2} are integers and the fact that the integers are closed under addition.

Now we prove that for all natural numbers $k \geq 3k1$, F is even. The base case, $k = 1$, is that $F_3 = 2$ is even, which is clear.

For the induction step, we have that $F_n = F_{n-1} + F_{n-2} = 2F_{n-2} + F_{n-3}$. Or that $F_{3k+3} = 2F_{3k+1} + F_{3k}$.

By the induction hypothese $F_{3k} = 2q$ for some q, and we have that $F_{3k+3} = 2(F_{3k+1} + q)$, which implies that it is even. Thus, by induction we have that all F_{3k} are even.

5. **Convergence of Series**

Use induction to prove that for all integers $n \geq 1$,

$$\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2.$$

Hint: Strengthen the induction hypothesis to $\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2 - \frac{1}{\sqrt{n}}$.

Solution: We use induction on n. The base case $n = 1$ is true because $1/3 < 1$. Assume the inequality holds for some $n \geq 1$. For $n+1$, by the inductive hypothesis, we have that

$$\sum_{k=1}^{n+1} \frac{1}{3k^{3/2}} = \sum_{k=1}^{n} \frac{1}{3k^{3/2}} + \frac{1}{3(n+1)^{3/2}} \leq 2 - \frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}}.$$

Thus, to prove our claim, it suffices to show that

$$-\frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}} \leq -\frac{1}{\sqrt{n+1}} \qquad (1)$$

This is a purely arithmetic problem and there are multiple ways to proceed.

Notice that to prove the inequality (**??**), it suffices to show that

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \geq \frac{1}{3(n+1)^{3/2}} = \frac{1}{3(n+1)\sqrt{n+1}},$$

which is equivalent to showing that

$$\frac{\sqrt{n+1}}{\sqrt{n}} - 1 = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} \geq \frac{1}{3(n+1)}.$$

So we want to show

$$\frac{\sqrt{n+1}}{\sqrt{n}} \geq \frac{1}{3(n+1)} + 1 = \frac{3n+4}{3n+3},$$

and squaring both sides means this is equivalent to

$$\frac{n+1}{n} \geq \frac{(3n+4)^2}{(3n+3)^2}.$$

At this point we cross-multiply, so we just need to show that

$$(n+1)(3n+3)^2 \geq n(3n+4)^2.$$

This is something that can be easily seen by expanding both sides and canceling terms, so we have shown Equation (**??**). This computation allows us to conclude that

$$\sum_{k=1}^{n+1} \frac{1}{3k^{3/2}} = \sum_{k=1}^{n} \frac{1}{3k^{3/2}} + \frac{1}{3(n+1)^{3/2}} \leq 2 - \frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}} \overset{(1)}{\leq} 2 - \frac{1}{\sqrt{n+1}},$$

where we have used equation (**??**) for the last inequality. This concludes the induction.

6. **Elephant Mosquito Paradox**

Claim: The weight of an elephant equals the weight of a mosquito.

Proof: Let x be the weight of an elephant, and y that of a mosquito. Call the sum of the two weights 2v, so that

$$x+y=2v$$

From this equation we can obtain two more.

$$x-2v=-y, x=-y+2v$$

Multiplying those together, we get

$$x^2-2vx=y^2-2vy$$

Add v^2 to both sides.

$$x^2-2vx+v^2=y^2-2vy+v^2$$
$$(x-v)^2=(y-v)^2$$

Taking square roots, we get

$$x-v=y-v$$

From this we conclude: $x=y$. That is, the elephant's weight (x) equals the mosquito's weight (y). Q.E.D. What is wrong here? You only need to find one wrong step, but identify all the wrong steps if you find more than one.

Solution: When taking the square root of both sides, we have to add absolute value bars to both sides. Instead of $x-v=y-v$, we should then have

$$|x-v|=|y-v|$$

. Intuitively, v is the average, given that $v=\frac{x+y}{2}$. As a result, it is clear that both x and y are the same distance away from z. From there, we can see that the trivial statement $|x-z|=|y-z|$ is just misconstrued $x-z=y-z$.