Discussion 2B

1. **Make it Stronger**

   Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that
   $$a_n \leq 3^{2^n}$$
   for every natural number n.

   (a) Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply $a_n \leq 3^{2^n}$? Show why this does not work.

   (b) Try to instead prove the statement $a_n \leq 3^{2^n-1}$ using induction. Does this statement imply what you tried to prove in the previous part?

2. **Well-Ordering Principle**

   In this question, we will go over how the well-ordering principle can be derived from (strong) induction. Remember the well-ordering principle states the following:

   For every non-empty subset S of the set of natural numbers $\mathbb{N}$, there is a smallest element $x \in S$; i.e.
   $$\exists x : \forall y \in S : x \leq y$$

   (a) What is the significane of S being non-empty? Does WOP hold without it? Assuming that S is not empty is equivalent to saying that there exists some number z in it.

   (b) Induction is always stated in terms of a property that can only be a natural number. What should the induction be based on? The length of the set S? The number x? The number y? The number z?

   (c) Now that the induction variable is clear, state the induction hypothesis. Be very precise. Do not leave out dangling symbols other than the induction variable. Ideally you should be able to write this in mathematical notation.

   (d) Verify the base case. Note that your base case does not just consist of a single set S.

   (e) Now prove that the induction works, by writing the inductive step.

   (f) What should you change so that the proof works by simple induction (as opoosed to strong induction)?

3. **Stable Marriage**

   The following questions refer to stable marriage instances with n men and n women, answer True/False or provide an expression as requested.

(a) For $n = 2$, or any 2-men, 2 woman stable marriage instance, man A has the same optimal and pessimal woman. (True or False)

(b) In any stable marriage instance, in the pairing in the TMA there is some man who gets his favorite woman (the first women on his preference list.) (True or False.)

(c) In any stable marriage instance with n men and women, if every man has a different favorite woman, a different second favorite, a different third, and so on, and every woman has the same preference list, how many days does it take for TMA to finish? (Form of Answer: An expression that may contain n.)

(d) Consider a stable marriage instance with n men and n women, and where all men have the same preference list, and all women have different favorites, and different second men, and so on. How many days does the TMA take to finish? (Form of Answer: An expression that may contain n)

(e) It is possible for a stable pairing to have a man A and a woman 1 be paired if A is 1's least preferred choice and 1 is A's least preferred choice. (True or False)

(f) It is possible for a stable pairing to have two couples where each person is paired with their lowest possible choice. (True or False)

(g) If there is a pairing, P, that consists of only pairs from man and woman optimal pairings, then it must be stable. In other words, if every pair in P is a pair either in the man optimal or the woman optimal pairing then P is stable. (True or false.)

4. **Pairing Up**

Prove that for every even $n \geq 2$, there exists an instance of the stable marriage problem with n men and n women such that the instance has at least $2^{n/2}$ distinct stable matchings.

5. **Good, Better, Best**

In a particular instance of the stable marriage problem with n men and n women, it turns out that there are exactly three distinct stable matchings, M_1, M_2, and M_3. Also, each man m has a different partner in the three matchings. Therefore each man has a clear preference ordering of the three matchings (according to the ranking of his partners in his preference list). Now, suppose for man m_1, this order is $M_1 > M_2 > M_3$.

Prove that every man has the same preference ordering $M_1 > M_2 > M_3$.