1. **Make it Stronger**

Suppose that the sequence $a_1, a_2, \dots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{2^n}$$

for every natural number $n$.

(a) Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply $a_n \leq 3^{2^n}$? Show why this does not work.

**Solution:** Try to prove that for every $n \geq 1$, we have $a_n \leq 3^{2^n}$ by induction.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2^1} = 9$.

Inductive Step: Assuming the statement is true for an $n$, we have

$$a_{n+1} = 3a_n^2 \leq 3(3^{2^n})^2 = 3 \times 3^{2 \times 2^n} = 3 \times 3^{2^{n+1}} = 3^{2^{n+1}+1}.$$

However, what we wanted was to get an inequality of the form: $a_{n+1} \leq 3^{2^{n+1}}$. There is an extra $+1$ in the exponent of what we derived.

(b) Try to instead prove the statement $a_n \leq 3^{2^n - 1}$ using induction. Does this statement imply what you tried to prove in the previous part?

**Solution:** This time the induction works.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2-1} = 3$.

Inductive Step: Assuming the hypothesis holds for $n$, we get

$$a_{n+1} = 3a_n^2 \leq 3 \times (3^{2^n - 1})^2 = 3 \times 3^{2 \times (2^n - 1)} = 3 \times 3^{2^{n+1} - 2} = 3^{2^{n+1} - 1}.$$

This is exactly the induction hypothesis for $n+1$. Note that for every $n \geq 1$, we have $2^n - 1 \leq 2^n$ and therefore $3^{2^n - 1} \leq 3^{2^n}$. This means that our modified hypothesis which we proved here does indeed imply what we wanted to prove in the previous part.

2. **Well-Ordering Principle**

In this question, we will go over how the well-ordering principle can be derived from (strong) induction. Remember the well-ordering principle states the following:

For every non-empty subset $S$ of the set of natural numbers $\mathbb{N}$, there is a smallest element $x \in S$; i.e.

$$\exists x : \forall y \in S : x \leq y$$

(a) What is the significane of $S$ being non-empty? Does WOP hold without it? Assuming that $S$ is not empty is equivalent to saying that there exists some number $z$ in it.

(b) Induction is always stated in terms of a property that can only be a natural number. What should the induction be based on? The length of the set $S$? The number $x$? The number $y$? The number $z$?

(c) Now that the induction variable is clear, state the induction hypothesis. Be very precise. Do not leave out dangling symbols other than the induction variable. Ideally you should be able to write this in mathematical notation.

(d) Verify the base case. Note that your base case does not just consist of a single set $S$.

(e) Now prove that the induction works, by writing the inductive step.

(f) What should you change so that the proof works by simple induction (as opoosed to strong induction)?

**Solution:** If $S$ is not empty, then WOP does not hold obviously. The significance is that you can always take a number out of it and start from there.

The induction is based on $z$. Formally the induction hypothesis in terms of $z$ would be:

**Hypothesis:** For all sets $S$ that contain $z$, the set $S$ contains a smallest element.

**Base Case:** For $z = 1$, the claim is true. Because we can take $x = 1$ as the smallest element and for all $y \in S$ we have $y \in \mathbb{N}$, and therefore $y \geq 1 = x$.

**Inductive Step:** Let $S$ be a set that contains $z$. If $z$ is the smallest element, we are done. Otherwise there exists $y \in S$, such that $y < z$. But now, by the induction hypothesis for $y$, we know that $S$ contains a smallest element.

In order to make the proof work without strong induction, one can modify the induction hypothesis in the following way: **Hypothesis (in terms of $z$)** For all sets $S$ that contain a number $z'$ such that $z' \leq z$, the set $S$ contains a smallest element.

This makes the proof work, simply because when we find a smaller element than $z$, we know that it is smaller than or equal to $z - 1$. So instead of going back all the way down to that element, we can just take one step back and appeal to the induction hypothesis for $z - 1$.

3. **Stable Marriage**

The following questions refer to stable marriage instances with $n$ men and $n$ women, answer True/False or provide an expression as requested.

(a) For $n = 2$, or any 2-men, 2 woman stable marriage instance, man A has the same optimal and pessimal woman. (True or False)

**Solution: False.** This says there is only one stable pairing. But preference list for man A is $(1,2)$ and for man B is $(2,1)$ and preference list for woman 1 is $(B,A)$ and woman 2 is $(A,B)$ produce different male and female optimal pairings.

(b) In any stable marriage instance, in the pairing in the TMA there is some man who gets his favorite woman (the first women on his preference list.) (True or False.)

**Solution: False.** Let man $A$ have preference list $(1,3,2)$, $B$ have $(1,2,3)$, and $C$ have $(2,1,3)$. We develop a "cyclic" chain of preferences, causing $A$ to displace $B$ to displace $C$ who then displaces $A$.

i. If woman 1 prefers $A$ over $B$, she puts $A$ on a string and rejects $B$.

ii. $B$ does not get his favorite and proposes to 2, who prefers $B$ over $C$ and thus rejects $C$.

iii. $C$ does not get his favorite and proposes to 1, who prefers $C$ over $A$ and thus rejects $A$.

Thus, $A$ also does not get his favorite, and no man gets his favorite.

(c) In any stable marriage instance with $n$ men and women, if every man has a different favorite woman, a different second favorite, a different third, and so on, and every woman has the same preference list, how many days does it take for TMA to finish? (Form of Answer: An expression that may contain $n$.)

**Solution: 1**.
On the first day every woman gets a proposal since each man has a different woman in their first position. The algorithm terminates.

(d) Consider a stable marriage instance with $n$ men and $n$ women, and where all men have the same preference list, and all women have different favorites, and different second men, and so on. How many days does the TMA take to finish? (Form of Answer: An expression that may contain $n$)

**Solution: n**.
Every man proposes to their common favorite. One man is kept on the string. The rest propose to the second. And so one. After each day, a new woman gets a man on a string. After $n$ days, we finish. Note: that the women's preference list were irrelevant.

(e) It is possible for a stable pairing to have a man A and a woman 1 be paired if A is 1's least preferred choice and 1 is A's least preferred choice. (True or False)

**Solution: True.**
A and 1 are respectively all the women's and men's least favorite.

(f) It is possible for a stable pairing to have two couples where each person is paired with their lowest possible choice. (True or False)

**Solution: False.**
Just consider the two couples. The man from the first and the woman from the other prefer each other, thus they form a rogue couple.

(g) If there is a pairing, $P$, that consists of only pairs from man and woman optimal pairings, then it must be stable. In other words, if every pair in $P$ is a pair either in the man optimal or the woman optimal pairing then $P$ is stable. (True or false.)

**Solution: False**.
Consider a woman who is matched to her pessimal partner and a man who is matched to his pessimal partner. They may well like each other.

An example is as follows.

Men's preference list
A: $1 > \ldots > 2$
B: $2 > \ldots > 1$
C: $3 > \ldots > 4$
D': $4 > \ldots > 3$

Women's preference list
1: $B > \ldots > A$
2: $A > \ldots > B$
3: $D > \ldots > C$
4: $C > \ldots > D$

Men's first choices = women's last choices and vice versa.
men-optimal: (A,1), (B,2), (C,3), (D,4)
women-optimal: (B,1), (A,2), (D,3), (C,4)
our pairing: (A,1), (B,2), (D,3), (C,4) and (C,1) is a rouge couple.

4. **Pairing Up**

Prove that for every even $n \geq 2$, there exists an instance of the stable marriage problem with $n$ men and $n$ women such that the instance has at least $2^{n/2}$ distinct stable matchings. **Solution:** To prove that there exists such a stable marriage instance for any even $n \geq 2$, we just need to show how to construct such an instance.

The idea here is that we can create pairs of men and pairs of women: pair up man $i$ and $i+1$ into a pair and woman $i$ and $i+1$ into a pair (you might come to this idea since we are asked to prove this for *even* $n$).

For $n$, we have $n/2$ pairs. Choose the preference lists such that the $k$th pair of men rank the $k$th pair of women just higher than the $(k+1)$th pair of women (the pairs wrap around from the last pair to the first pair), and the $k$th pair of women rank the $k$th pair of men just higher than the $(k+1)$th pair of men. Within each pair of pairs $(m,m')$ and $(w,w')$, let $m$ prefer $w$, let $m'$ prefer $w'$, let $w$ prefer $m'$, and let $w'$ prefer $m$. It might help to draw out an example on the board with arrows denoting preferences.

Each match will have men in the $k$th pair paired to women in the $k$th pair for $1 \leq k \leq n/2$.

A man $m$ in pair $k$ will never form a rogue couple with any woman $w$ in pair $j \neq k$. If $j > k$, then $w$ prefers her current partner in the $j$th pair to $m$. If $j < k$, then $m$ prefers his current partner in the $k$th pair to $w$. Then a rogue couple could only exist in the same pair - but this is impossible since exactly one of either $m$ or $w$ must be married to their preferred choice in the pair.

Since each man in pair $k$ can be stably married to either woman in pair $k$, and there are $n/2$ total pairs, the number of stable matchings is $2^{n/2}$.

5. **Good, Better, Best**

In a particular instance of the stable marriage problem with $n$ men and $n$ women, it turns out that there are exactly three distinct stable matchings, $M_1$, $M_2$, and $M_3$. Also, each man $m$ has a different partner in the three matchings. Therefore each man has a clear preference ordering of the three matchings (according to the ranking of his partners in his preference list). Now, suppose for man $m_1$, this order is $M_1 > M_2 > M_3$.

Prove that every man has the same preference ordering $M_1 > M_2 > M_3$.

**Solution:** In class, you were given the traditional propose-and-reject algorithm, which was guaranteed to produce a male-optimal matching. By switching men's and women's roles, you would be guaranteed to produce a female-optimal matching, which, by a lemma from class, would also be male-pessimal. By the very fact that these algorithms exist and have been proven to work in this way, you're guaranteed that a male-optimal and a male-pessimal matching always exist.

Since there are only three matchings in this particular stable matching instance, we thus know that one of them must be male-optimal and one must be male-pessimal. Since $m_1$ prefers $M_1$ above the other stable matchings, only that one can be male-optimal by definition of male-optimality. Similarly, since $m_1$ prefers $M_3$ the least, it must be the male-pessimal. Therefore, again from definitions of optimality/pessimality, since each men have different matches in the three stable matchings, they *must* strictly prefer $M_1$ to both of the others, and they *must* like $M_3$ strictly less than both of the others. Thus, each man's preference order of stable matchings must be $M_1, M_2, M_3$.