CS 70 Discrete Mathematics and Probability Theory
Fall 2016 Seshia and Walrand **Discussion 3A**

1. **Odd Degree Vertices**

   Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

   Prove the claim above using:

   (i) Direct proof (e.g., counting the number of edges in G)

   (ii) Induction on $m = |E|$ (number of edges)

   (iii) Induction on $n = |V|$ (number of vertices)

   (iv) Well-ordering principle

2. **Build-up Error?**

   What is wrong with the following "proof"?

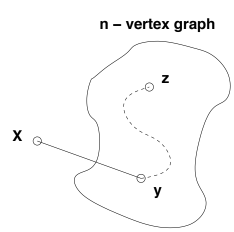
   False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

   Proof: We use induction on the number of vertices $n \geq 1$.

   Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

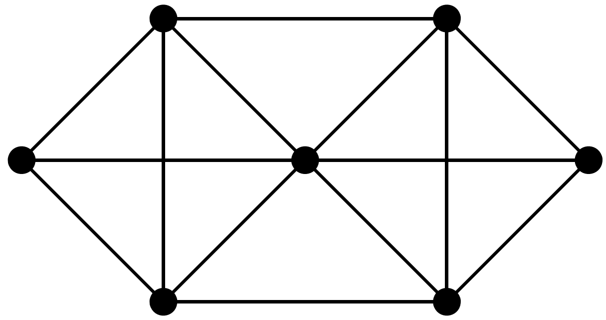
   Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

   Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



   All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x, y\}$ to the path from y to z. This proves the claim for $n+1$.

3. **Eulerian Tour and Eulerian Walk**



(a) Is there an Eulerian tour in the graph above?

(b) Is there an Eulerian walk in the graph above?

(c) What is the condition that there is an Eulerian walk in an undirected graph?

4. **Bipartite Graph**

Consider an undirected bipartite graph with two disjoint sets L, R. Prove that a bipartite graph has no cycles of odd length.

5. **Leaves in a Tree**

A *leaf* in a tree is a vertex with degree 1.

(a) Prove that every tree on $n \geq 2$ vertices has at least two leaves.

(b) What is the maximum number of leaves in a tree with $n \geq 3$ vertices?