1. **Odd Degree Vertices**

Claim: Let $G = (V,E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G)
(ii) Induction on $m = |E|$ (number of edges)
(iii) Induction on $n = |V|$ (number of vertices)
(iv) Well-ordering principle

Solution: Let $V_{\text{odd}}(G)$ denote the set of vertices in G that have odd degree. We prove that $|V_{\text{odd}}(G)|$ is even.

(i) Let d_v denote the degree of vertex v (so $d_v = |N_v|$, where N_v is the set of neighbors of v). Observe that

$$\sum_{v \in V} d_v = 2m$$

because every edge is counted exactly twice when we sum the degrees of all the vertices. Now partition V into the odd degree vertices $V_{\text{odd}}(G)$ and the even degree vertices $V_{\text{odd}}(G)^{\complement}$, so we can write

$$\sum_{v \in V_{\text{odd}}(G)} d_v = 2m - \sum_{v \notin V_{\text{odd}}(G)} d_v.$$

Both terms in the righthand side above are even ($2m$ is even, and each term d_v is even because we are summing over even degree vertices $v \notin V_{\text{odd}}(G)$). So for the lefthand side $\sum_{v \in V_{\text{odd}}(G)} d_v$ to be even, we must have an even number of terms, since each term in the summation is odd. Therefore, there must be an even number of odd-degree vertices, namely, $|V_{\text{odd}}(G)|$ is even.

(ii) We use induction on $m \geq 0$.

Base case $m = 0$: If there are no edges in G, then all vertices have degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with m edges.

Inductive step: Let G be a graph with $m+1$ edges. Remove an arbitrary edge $\{u,v\}$ from G, so the resulting graph G' has m edges. By the inductive hypothesis, we know $|V_{\text{odd}}(G')|$ is even. Now add the edge $\{u,v\}$ to get back the original graph G. Note that u has one more edge in G than it does in G', so $u \in V_{\text{odd}}(G)$ if and only if $u \notin V_{\text{odd}}(G')$.

Similarly, $v \in V_{\text{odd}}(G)$ if and only if $v \notin V_{\text{odd}}(G')$. The degrees of all other vertices are unchanged in going from G' to G. Therefore,

$$V_{\text{odd}}(G) = \begin{cases} V_{\text{odd}}(G') \cup \{u,v\} & \text{if } u,v \notin V_{\text{odd}}(G') \\ V_{\text{odd}}(G') \setminus \{u,v\} & \text{if } u,v \in V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{u\}) \cup \{v\} & \text{if } u \in V_{\text{odd}}(G'), v \notin V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{v\}) \cup \{u\} & \text{if } u \notin V_{\text{odd}}(G'), v \in V_{\text{odd}}(G') \end{cases}$$

so we see that $|V_{\text{odd}}(G)| - |V_{\text{odd}}(G')| \in \{-2,0,2\}$. Since $|V_{\text{odd}}(G')|$ is even, we conclude $|V_{\text{odd}}(G)|$ is also even.

(iii) We use induction on $n \geq 1$.

Base case $n = 1$: If G only has 1 vertex, then that vertex has degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with n vertices.

Inductive step: Let G be a graph with $n+1$ vertices. Remove a vertex v and all edges adjacent to it from G. The resulting graph G' has n vertices, so by the inductive hypothesis, $|V_{\text{odd}}(G')|$ is even. Now add the vertex v and all edges adjacent to it to get back the original graph G. Let $N_v \subseteq V$ denote the neighbors of v (i.e., all vertices adjacent to v). Among the neighbors N_v, the vertices in the intersection $A = N_v \cap V_{\text{odd}}(G')$ had odd degree in G', so they now have even degree in G. On the other hand, the vertices in $B = N_v \cap V_{\text{odd}}(G')^{\complement}$ had even degree in G', and they now have odd degree in G. The vertex v itself has degree $|N_v|$, so $v \in V_{\text{odd}}(G)$ if and only if $|N_v|$ is odd. We now consider two cases:

i. Suppose $|N_v|$ is even, so $v \notin V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B$$

so $|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B|$. Note that A and B are disjoint and their union equals N_v, so $|A| + |B| = |N_v|$. Therefore, we can write $|V_{\text{odd}}(G)|$ as

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| + |N_v| - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is even by assumption.

ii. Suppose $|N_v|$ is odd, so $v \in V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B \cup \{v\}$$

so, again using the relation $|A| + |B| = |N_v|$, we can write

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B| + 1 = |V_{\text{odd}}(G')| + (|N_v| + 1) - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is odd by assumption.

This completes the inductive step and the proof.

Note how this proof is more complicated than the proof in part (i), even though they are both using induction. This tells you that choosing the right variable to induct on can simplify the proof.

(iv) Here we give a well-ordering proof using the number of edges m as the notion of "size" of G, so this is equivalent to the proof in part (i) using induction on m. (You can also try to give a well-ordering proof using n as the size of G.)

Suppose the contrary that the claim is false for some graphs. This means the set M is not empty, where M is the set of $m \in \mathbb{N}$ for which there exists a graph G with m edges that is a counterexample to the claim. Thus, we have a nonempty subset M of $\mathbb{N}$, so by the well-ordering principle, M has a smallest element m'. Note that $m' > 0$, since the claim is true for all graphs with 0 edges.

Let G be a graph with m' edges for which the claim is false, i.e., $|V_{\text{odd}}(G)|$ is odd (here we know such a G must exist from the definition of $m' \in M$). Remove one edge from G to obtain a smaller graph G' with $m' - 1$ edges (here we need $m' \geq 1$, which we have seen above). By our choice of m' as the smallest element of M, we know that $m' - 1 \notin M$, so the claim holds for G', namely, $|V_{\text{odd}}(G')|$ is even. Now add the removed edge to get back G. By the same argument as in the inductive step in part (i), this implies that $|V_{\text{odd}}(G)|$ is also even, a contradiction.

2. **Build-up Error?**

What is wrong with the following "proof"?

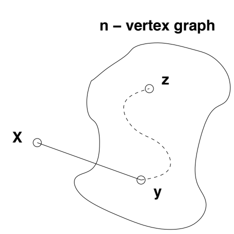
False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$.

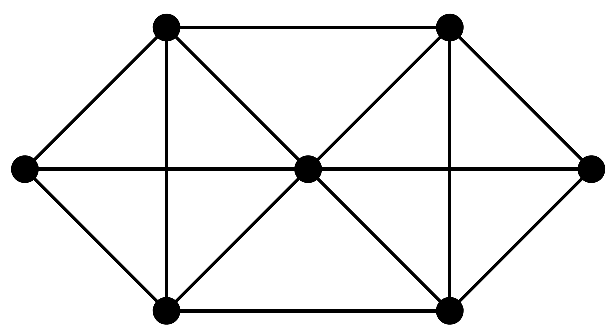
Solution: The mistake is in the argument that *"every $(n+1)$-vertex graph with minimum degree 1 can be obtained from an n-vertex graph with minimum degree 1 by adding 1 more vertex."* Instead of starting by considering an arbitrary $(n+1)$-vertex graph, this proof only considers an $(n+1)$-vertex graph that you can make by starting with an n-vertex graph with minimum degree 1. As a counterexample, consider a graph on four vertices $V = \{1,2,3,4\}$ with two edges $E = \{\{1,2\},\{3,4\}\}$. Every vertex in this graph has degree 1, but there is no way to build this 4-vertex graph from a 3-vertex graph with minimum degree 1.

More generally, this is an example of *build-up error* in proof by induction. Usually this arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others, such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a *"shrink down, grow back"* process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds.

Let's see what would have happened if we'd tried to prove the claim above by this method. In the inductive step, we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$-vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh! The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck —– and properly so, since the claim is false!

3. **Eulerian Tour and Eulerian Walk**



(a) Is there an Eulerian tour in the graph above?

Solution: No. Two vertices have odd degree.

(b) Is there an Eulerian walk in the graph above?

Solution: Yes. One of the two vertices with odd degree must be the first vertex, and the other one must be the last vertex.

(c) What is the condition that there is an Eulerian walk in an undirected graph?

Solution: An undirected graph has an Eulerian walk if and only if it is connected (except for isolated vertices) and at most two vertices have odd degree.

Note: There is no graph with only one odd degree vertex.

4. **Bipartite Graph**

Consider an undirected bipartite graph with two disjoint sets L, R. Prove that a bipartite graph has no cycles of odd length.

Solution: Let us start traveling the cycle from a node n_0 in L. Since each edge in the graph connects a vertex in L to one in R, the 1st edge in the set connects our start node n_0 to the a node n_1 in R. The 2nd edge in the cycle must connect n_1 to a node n_2 in L. Continuing on, the $(2k+1)$-th edge connects node n_{2k} in L to node n_{2k+1} in R, and the $2k$-th edge connects node n_{2k-1} in R to node n_{2k} in L. Since only even numbered edges connect to vertices in L, and we started our cycle in L, the cycle must edge with an even number of edges.

5. **Leaves in a Tree**

A *leaf* in a tree is a vertex with degree 1.

(a) Prove that every tree on $n \geq 2$ vertices has at least two leaves.

Solution: We give a direct proof. Consider the longest path $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}$ between two vertices $x = v_0$ and $y = v_k$ in the tree (here the length of a path is how many edges it uses, and if there are multiple longest paths then we just pick one of them). We claim that x and y must be leaves. Suppose the contrary that x is not a leaf, so it has degree at least two. This means x is adjacent to another vertex z different from v_1. Observe that z cannot appear in the path from x to y that we are considering, for otherwise there would be a cycle in the tree. Therefore, we can add the edge $\{z, x\}$ to our path to obtain a longer path in the tree, contradicting our earlier choice of the longest path. Thus, we conclude that x is a leaf. By the same argument, we conclude y is also a leaf.

The case when a tree has only two leaves is called the *path graph*, which is the graph on $V = \{1, 2, \ldots, n\}$ with edges $E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}\}$.

(b) What is the maximum number of leaves in a tree with $n \geq 3$ vertices?

Solution: We claim the maximum number of leaves is $n-1$. This is achieved when there is one vertex that is connected to all other vertices (this is called the *star graph*).

We now show that a tree on $n \geq 3$ vertices cannot have n leaves. Suppose the contrary that there is a tree on $n \geq 3$ vertices such that all its n vertices are leaves. Pick an arbitrary vertex x, and let y be its unique neighbor. Since x and y both have degree 1, the vertices x, y form a connected component separate from the rest of the tree, contradicting the fact that a tree is connected.