1. **Trees**

   Recall that a *tree* is a connected acyclic graph (graph without cycles). In the note, we presented a few other definitions of a tree, and in this problem, we will prove two fundamental properties of a tree, and derive two definitions of a tree we learn from lecture note based on these properties. Let's start with the properties:

   (a) Prove that any pair of vertices in a tree are connected by exactly one (simple) path.

   (b) Prove that adding any edge between two vertices of a tree creates a simple cycle.

   Now you will show that if a graph satisfies either of these two properties then it must be a tree:

   (c) Prove that if every pair of vertices in a graph are connected by exactly one simple path, then the graph must be a tree.

   (d) Prove that if the graph has no simple cycles and has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

2. **Hypercubes**

   The vertex set of the n-dimensional hypercube $G = (V,E)$ is given by $V = \{0,1\}^n$, where recall $\{0,1\}^n$ denotes the set of all n-bit strings. There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

   (a) Draw 1-, 2-, and 3-dimensional hypercubes.

   (b) Show that the edges of an n-dimensional hypercube can be colored using n colors so that no pair of edges sharing a common vertex have the same color.

   (c) Show that the vertices of an n-dimensional hypercube can be colored using 2 colors so that no pair of adjacent vertices have the same color. (This is equivalent to showing that a hypercube is *bipartite*: the vertices can be partitioned into two groups (according to color) so that every edge goes between the two groups.)

3. **Planarity**

   Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

4. **Graph Coloring**

   Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

5. **Modular decomposition of modular arithmetic**

   Complex systems are always broken down into simpler modules. In this problem you will learn how this might be done in modular arithmetic.

   (a) Write down the addition and multiplication table for modular-6 arithmetic (the rows and columns should be labeled $0,1,2,3,4,5$).

   (b) Each number $0,1,2,3,4,5$ has a remainder mod 2 and a remainder mod 3. For each number write down the pair (x,y) where x is its remainder mod 2 and y is its remainder mod 3. Obviously $0 \leq x \leq 1$ and $0 \leq y \leq 2$. Out of all possible pairs (x,y), where $0 \leq x \leq 1$ and $0 \leq y \leq 2$, how many times do you see each pair appear?

   (c) Again write down the addition and multiplication table you wrote in part 1, but this time replace each number with its corresponding pair (when a number appears as a row/column label and also when it appears somewhere in the table). Describe how one can add or multiply two pairs without looking at the original numbers.

6. **Does it Exist?**

   Can you find a number that is a perfect square and is a multiple of 2 but not a multiple of 4? Either give such a number or prove that no such number exists.