CS 70 Fall 2016 — Discrete Mathematics and Probability Theory — Seshia and Walrand — Discussion 3B

1. **Trees**

Recall that a *tree* is a connected acyclic graph (graph without cycles). In the note, we presented a few other definitions of a tree, and in this problem, we will prove two fundamental properties of a tree, and derive two definitions of a tree we learn from lecture note based on these properties. Let's start with the properties:

(a) Prove that any pair of vertices in a tree are connected by exactly one (simple) path.

Solution: Pick any pair of vertices x, y. We know there is a path between them since the graph is connected. We will prove that this path is unique by contradiction: Suppose there are two distinct paths from x to y. At some point (say at vertex a) the paths must diverge, and at some point (say at vertex b) they must reconnect. So by following the first path from a to b and the second path in reverse from b to a we get a cycle. This gives the necessary contradiction.

(b) Prove that adding any edge between two vertices of a tree creates a simple cycle.

Solution: Pick any pair of vertices x, y not connected by an edge. We prove that adding the edge $\{x,y\}$ will create a simple cycle. From part (a), we know that there is a unique path between x and y. Therefore, adding the edge $\{x,y\}$ creates a simple cycle obtained by following the path from x to y, then following the edge $\{x,y\}$ from y back to x.

Now you will show that if a graph satisfies either of these two properties then it must be a tree:

(c) Prove that if every pair of vertices in a graph are connected by exactly one simple path, then the graph must be a tree.

Solution: Assume we have a graph with the property that there is a unique simple path between every pair of vertices. We will show that the graph is a tree, namely, it is connected and acyclic. First, the graph is connected because every pair of vertices is connected by a path. Moreover, the graph is acyclic because there is a unique path between every pair of vertices. More explicitly, if the graph has a cycle, then for any two vertices x, y in the cycle there are at least two simple paths between them (obtained by going from x to y through the right or left half of the cycle), contradicting the uniqueness of the path. Therefore, we conclude the graph is a tree.

(d) Prove that if the graph has no simple cycles and has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

Solution: Assume we have a graph with no simple cycles, but adding any edge will create a simple cycle. We will show that the graph is a tree. We know the graph is acyclic because it has no simple cycles. To show the graph is connected, we prove that any pair of vertices x, y are connected by a path. We consider two cases: If $\{x,y\}$ is an edge, then clearly there is a path from x to y. Otherwise, if $\{x,y\}$ is not an edge, then by assumption, adding the edge $\{x,y\}$ will create a simple cycle. This means there is a simple path from x to y obtained by removing the edge $\{x,y\}$ from this cycle. Therefore, we conclude the graph is a tree.

2. **Hypercubes**

The vertex set of the n-dimensional hypercube $G = (V,E)$ is given by $V = \{0,1\}^n$, where recall $\{0,1\}^n$ denotes the set of all n-bit strings. There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes.

Solution: Check the graphs on Page 9, Lecture Note 18.

(b) Show that the edges of an n-dimensional hypercube can be colored using n colors so that no pair of edges sharing a common vertex have the same color.

Solution: Consider each edge that changes the i^{th} bit for some $i \le n$. Every vertex touches exactly one of these edges, because there is exactly one way to change the i^{th} bit in any bitstring. Coloring each of these edges color i ensures that each vertex will then be adjacent to n differently colored edges, since there are n different bits to change, and no two edges representing bit changes on different bits have the same color.

(c) Show that the vertices of an n-dimensional hypercube can be colored using 2 colors so that no pair of adjacent vertices have the same color. (This is equivalent to showing that a hypercube is *bipartite*: the vertices can be partitioned into two groups (according to color) so that every edge goes between the two groups.)

Solution: Consider the vertices with an even number of 0 bits and the vertices with an odd number of 0 bits. Each vertex with an even number of 0 bits is adjacent only to vertices with an odd number of 0 bits, since each edge represents a single bit change (either a 0 bit is added by flipping a 1 bit, or a 0 bit is removed by flipping a 0 bit). By coloring the vertices with an even number of 0 bits color 0 and vertices with an odd number of 0 bits color 1, no two adjacent vertices will share a color.

3. **Planarity**

Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Solution:

Assume G is planar. Take 5 vertices, they cannot form K_5, so some pair v_1, v_2 have no edge between them. The remaining five vertices of G cannot form K_5 either, so there is a second pair v_3, v_4 that have no edge between them. Now consider v_1, v_2 and any other three vertices v_5, v_6, v_7. Since $v_1 v_2$ is not an edge, by property T it must be that $v_1 v$ and $v_2 v$ where $v \in \{v_5, v_6, v_7\}$ are edges. Similarly for $v_3, v_4, v_3 v$ and $v_4 v$ where $v \in v_5, v_6, v_7$ are edges. So now v_1, v_2, v_3 on one side and v_5, v_6, v_7 on the other form an instance of $K_{3,3}$. Contradiction.

4. **Graph Coloring**

   Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

   Solution: The natural way to try to prove this theorem is to use induction on k. Unfortunately, this approach leads to disaster. It is not that it is impossible, just that it is extremely painful and would ruin your week if you tried it on an exam. When you encounter such a disaster when using induction on graphs, it is usually best to change what you are inducting on. In graphs, typical good choices for the induction parameter are n, the number of nodes, or e, the number of edges.

   We use induction on the number of vertices in the graph, which we denote by n. Let $P(n)$ be the proposition that an n-vertex graph with maximum degree at most k is $(k+1)$-colorable.

   Base Case $n = 1$: A 1-vertex graph has maximum degree 0 and is 1-colorable, so $P(1)$ is true.

   Inductive Step: Now assume that $P(n)$ is true, and let G be an $(n+1)$-vertex graph with maximum degree at most k. Remove a vertex v (and all edges incident to it), leaving an n-vertex subgraph,H. The maximum degree of H is at most k, and so H is $(k+1)$-colorable by our assumption $P(n)$. Now add back vertex v. We can assign v a color (from the set of $k+1$ colors) that is different from all its adjacent vertices, since there are at most k vertices adjacent to v and so at least one of the $k+1$ colors is still available. Therefore, G is $(k+1)$-colorable. This completes the inductive step, and the theorem follows by induction.

5. **Modular decomposition of modular arithmetic**

   Complex systems are always broken down into simpler modules. In this problem you will learn how this might be done in modular arithmetic.

   (a) Write down the addition and multiplication table for modular-6 arithmetic (the rows and columns should be labeled $0, 1, 2, 3, 4, 5$).

   Solution: The answer is found below.

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

(b) Each number $0,1,2,3,4,5$ has a remainder mod 2 and a remainder mod 3. For each number write down the pair (x,y) where x is its remainder mod 2 and y is its remainder mod 3. Obviously $0 \leq x \leq 1$ and $0 \leq y \leq 2$. Out of all possible pairs (x,y), where $0 \leq x \leq 1$ and $0 \leq y \leq 2$, how many times do you see each pair appear?

Solution: The answer can be found in the following table.

Number	mod 2	mod 3	Pair
0	0	0	(0, 0)
1	1	1	(1, 1)
2	0	2	(0, 2)
3	1	0	(1, 0)
4	0	1	(0, 1)
5	1	2	(1, 2)

As it can be easily seen, no pair is repeated and every pair (x,y) with $0 \leq x \leq 1$ and $0 \leq y \leq 2$ appears exactly once.

(c) Again write down the addition and multiplication table you wrote in part 1, but this time replace each number with its corresponding pair (when a number appears as a row/column label and also when it appears somewhere in the table). Describe how one can add or multiply two pairs without looking at the original numbers.

Solution: The tables are re-written below.

+	(0, 0)	(1, 1)	(0, 2)	(1, 0)	(0, 1)	(1, 2)
(0, 0)	(0, 0)	(1, 1)	(0, 2)	(1, 0)	(0, 1)	(1, 2)
(1, 1)	(1, 1)	(0, 2)	(1, 0)	(0, 1)	(1, 2)	(0, 0)
(0, 2)	(0, 2)	(1, 0)	(0, 1)	(1, 2)	(0, 0)	(1, 1)
(1, 0)	(1, 0)	(0, 1)	(1, 2)	(0, 0)	(1, 1)	(0, 2)
(0, 1)	(0, 1)	(1, 2)	(0, 0)	(1, 1)	(0, 2)	(1, 0)
(1, 2)	(1, 2)	(0, 0)	(1, 1)	(0, 2)	(1, 0)	(0, 1)

×	(0, 0)	(1, 1)	(0, 2)	(1, 0)	(0, 1)	(1, 2)
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(1, 1)	(0, 0)	(1, 1)	(0, 2)	(1, 0)	(0, 1)	(1, 2)
(0, 2)	(0, 0)	(0, 2)	(0, 1)	(0, 0)	(0, 2)	(0, 1)
(1, 0)	(0, 0)	(1, 0)	(0, 0)	(1, 0)	(0, 0)	(1, 0)
(0, 1)	(0, 0)	(0, 1)	(0, 2)	(0, 0)	(0, 1)	(0, 2)
(1, 2)	(0, 0)	(1, 2)	(0, 1)	(1, 0)	(0, 2)	(1, 1)

One can simply add or multiply pairs by adding or multiplying each coordinate separately. We basically can take the addition and multiplication modulo 2 for the first coordinates and modulo 3 for the second coordinates.

6. **Does it Exist?**

Can you find a number that is a perfect square and is a multiple of 2 but not a multiple of 4? Either give such a number or prove that no such number exists.

Solution: No, such a number does not exist. In mathematical notation, we are asked to find a number x such that, for $n \in \mathbb{N}$, $4n+2 = x^2$. Taking mod 4 of both sides, we get $x^2 \equiv 2 \pmod{4}$. Does this number exist? We will determine this by cases.

We know that $x \pmod{4}$ can have 4 different values:

- $x \mod 4 = 0$. Then $x^2 \equiv 0 \pmod{4}$.
- $x \mod 4 = 1$. Then $x^2 \equiv 1 \pmod{4}$.
- $x \mod 4 = 2$. Then $x^2 \equiv 4 \equiv 0 \pmod{4}$.
- $x \mod 4 = 3$. Then $x^2 \equiv 9 \equiv 1 \pmod{4}$.

Therefore we can see it is impossible to find such a number; no matter what x is, x^2 will never be a multiple of 4 plus 2.