1. **No Equal Digits** How many 7-digit numbers have no two adjacent digits equal?

   Solution:

   We construct these numbers from left-to-right. We have 9 choices for the first digit (since it cannot be 0). Then, no matter what we choose for the first digit, we have 9 choices for the second digit (any digit except the one that we chose for the first digit). Similarly, for each subsequent digit, we have 9 choices, since each digit can be any of 0 through 9 as long as it does not match the previous digit. Therefore, there are 9 choices for every digit, and hence there are $9^7 = 4,782,969$ such numbers.

2. **Strings** What is the number of strings you can construct given:

   (a) n ones, and m zeroes.

   (b) n_1 A's, n_2 B's and n_3 C's.

   (c) $n_1, n_2, \ldots, n_k$ respectively of k different letters.

   Solution:

   (a) $\binom{n+m}{n}$

   (b) $(n_1 + n_2 + n_3)!/(n_1! \cdot n_2! \cdot n_3!)$

   (c) $(n_1 + n_2 + \ldots + n_k)!/(n_1! \cdot n_2! \ldots n_k!)$.

3. **Palindromes** How many 5-digit palindromes are there? (A palindrome is a number that reads the same way forwards and backwards. For example, 27872 and 48484 are palindromes, but 28389 and 12541 are not.)

   Solution:

   We construct the number from left-to-right. We have 9 choices for the first digit (since it can't be 0), then 10 choices for the second digit, then 10 choices for the third digit. But now we're out of choices: the fourth digit must match the second, and the last digit must match the first. Therefore, there are 9 * 10 * 10 = 900 such numbers.

4. **Fruits** Suppose you want to buy n fruits, and you can buy 0 or more of any type. In how how many ways can you do that if:

   (a) There are apples and oranges at the market.

   (b) There are apples, oranges, and bananas at the market.

   (c) There are k kinds of fruits at the market.

Solution: This is a classic stars and bars problem.

(a) $n+1$

(b) $\binom{n+2}{2}$

(c) $\binom{n+k-1}{k-1}$.

5. **Combinatorial Proof III** Prove $\displaystyle\binom{2n}{n} = 2\binom{2n-1}{n-1}$

Solution:

LHS: Choose n elements from $2n$ elements.

RHS: Suppose we pick the last item, then we pick $n-1$ elements from the remaining $2n-1$. Suppose we don't pick the last item, then we pick n elements from the remaning $2n-1$. Notice that the two events are mutually exclusive, hence the number of ways on the RHS is $\binom{2n-1}{n-1} + \binom{2n-1}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n-1} = 2\binom{2n-1}{n-1}$.