1. **Probability Practice**

   (a) If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?

   (b) A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?

   (c) If five numbers are selected at random from the set $\{1,2,3,\ldots,20\}$, what is the probability that their minimum is larger than 5? (A number can be chosen more than once.)

Solution:

(a) $\frac{18!5!}{22!} = \frac{1}{1463}$. The 18! comes from 18 "units": 3 physics books, 8 engineering books, 6 biology books and 1 block of math books. The 5! comes from number of ways to arrange the 5 math books within the same block. 22! is just the total number of ways to arrange the books.

(b) $\left(\frac{2}{3}\right)^8 = \frac{256}{6561}$. This is just by independence.

(c) $\left(\frac{15}{20}\right)^5 = \frac{243}{1024}$. For a single number, we can choose 6,7...20, so 15 valid outcomes out of 20 total outcomes. Then apply independence as in part c.

2. **Unlikely events**

   (a) Toss a fair coin x times. What is the probability that you never get heads?

   (b) Roll a fair die x times. What is the probability that you never roll a six?

   (c) Suppose your weekly local lottery has a winning chance of $1/10^6$. You buy lottery from them for x weeks in a row. What is the probability that you never win?

   (d) How large must x be so that you get a head with probability at least 0.9? Roll a 6 with probability at least 0.9? Win the lottery with probability at least 0.9?

Solution:

(a) 0.5^x

(b) $(1-\frac{1}{6})^x$

(c) $(1-1/10^6)^x$

(d) i. For coin, want: $0.5^x \le 0.1$ so $x \ge \frac{\log 0.1}{\log 0.5} \approx 3.32$

ii. For die, want: $(5/6)^x \le 0.1$ so $x \ge \frac{\log 0.1}{\log 5/6} \approx 12.6$

iii. For the lottery, want: $(1 - 1/10^6)^x \le 0.1$ so $x \ge \frac{\log 0.1}{\log(1-1/10^6)} \approx 2 * 10^6$. (The answer is approximately equal to $\frac{\log 0.1}{-1/10^6}$, using the approximation for small values $(1-x) \approx e^{-x}$, where $x = \frac{1}{10^6}$.

3. **Flippin' Coins**

Suppose I have a biased coin, with outcomes H and T, with the probability of heads $Pr[H] = \frac{3}{4}$ and the probability of tails $Pr[T] = \frac{1}{4}$. Suppose I perform an experiment in which I toss the coin 3 times–an outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for my experiment?

(b) Which of the following are examples of *events*? Select all that apply.

i. $\{(H,H,T),(H,H),(T)\}$

ii. $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$

iii. $\{(T,T,T)\}$

iv. $\{(T,T,T),(H,H,H)\}$

v. $\{(T,H,T),(H,H,T)\}$

(c) What is the probability of the outcome H,H,T?

(d) What is the probability of the event that my outcome has exactly two heads?

Solution:

(a) $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

(b) i. No

ii. Yes

iii. Yes

iv. Yes

v. Yes

(c) $\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{64}$

(d) $\omega \in \{(H,H,T),(H,T,H),(T,H,H)\}$. The probability = $3 \cdot \frac{9}{64} = \frac{27}{64}$.

4. **Shooting Range**

You and your friend are at a shooting range. You ran out of bullets. Your friend still has two bullets left but magically lost his gun. Somehow you both agree to put the two bullets into your six-chambered revolver in successive order, spin the revolver, and then take turn shooting. Your first shot was a blank. You want your friend to shoot a blank too, should you spin the revolver again before you hand it to your friend?

Solution:

No, you shouldn't.

The first chamber fired was one of the four empty chambers. Since the bullets were placed in consecutive order, one of the empty chambers is followed by a bullet, and the other three empty chambers are followed by another empty chamber. So the probability that a bullet will be fired is 1/4.

If you spins the chamber again, the probability that a real bullet is shot would be 2/6, or 1/3, since there are two possible bullets that would be in firing position out of the six possible chambers that would be in position.