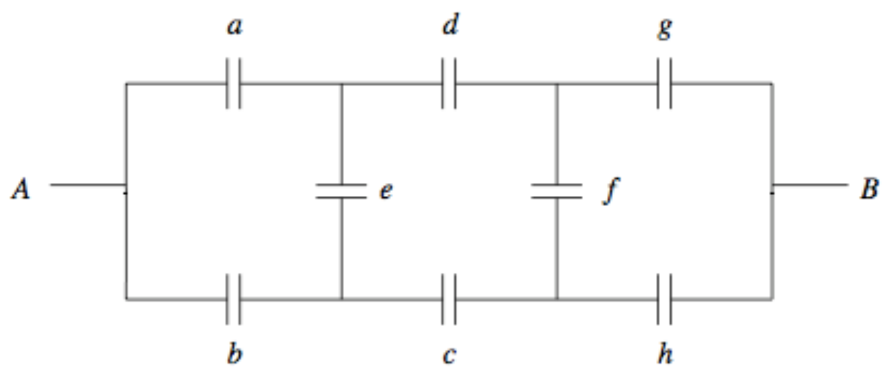
CS 70 Discrete Mathematics and Probability Theory
Fall 2016 Seshia and Walrand **Discussion 8b**

1. **Communication network**

   In the communication network shown below, link failures are independent, and each link has a probability of failure of p. Consider the physical situation before you write anything. A can communicate with B as long as they are connected by at least one path which contains only in-service links.



   (a) Given that exactly five links have failed, determine the probability that A can still communicate with B.

   (b) Given that exactly five links have failed, determine the probability that either g or h (*but not both*) is still operating properly.

   (c) Given that a, d and h have failed (but no information about the information of the other links), determine the probability that A can communicate with B.

   Solution:

   (a) There are only two paths of 3 links from A to B. And there are $\binom{8}{5}$ ways of the links messing up.

   So the probability is $\frac{2}{56} = \frac{1}{28}$.

   This is because every single case of exactly 5 links being down have the same probability. So it's a uniform distribution over all possibilities.

   (b) Fix g as down and h as working. There are $\binom{6}{4}$ ways to have 4 out of the remaining go down. Symmetric argument for h down and g up.

   So probability is $\frac{30}{56} = \frac{15}{28}$.

(c) We would just want the 4 on the only remaining path from A to B not to be down. The probability of this happening is $(1-p)^4$.

2. **Boy or Girl Paradox**

You know Mr. Smith has two children, at least one of whom is a boy. Assume that gender is independent and uniformly distributed, so for any child, the probability that they are a boy is the same as the probability they are a girl, which is $\frac{1}{2}$.

(a) What is the probability that both children are boys?

(b) Now suppose you knock on Mr. Smith's front door and you are greeted by a boy who you correctly deduce to be Mr. Smith's son. What is the probability that he has two boys? Compare your answer to the answer in part (a).

Solution:

(a) Let B_1 be the event that the first child is a boy, and B_2 be the event that the second child is a boy. We are asked to find $Pr[(B_1 \cap B_2)|(B_1 \cup B_2)]$:

$$\begin{aligned}
Pr[(B_1 \cap B_2)|(B_1 \cup B_2)] &= \frac{Pr[(B_1 \cap B_2) \cap (B_1 \cup B_2)]}{Pr[B_1 \cup B_2]} \\
&= \frac{Pr[B_1 \cap B_2]}{Pr[B_1] + Pr[B_2] - Pr[B_1 \cap B_2]} \\
&= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} + \frac{1}{2} - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} \\
&= \frac{1}{3}
\end{aligned}$$

Note: It is tempting to think that because the children's genders are independent, the probability of the second child being a boy given that the first is a boy is simply $\frac{1}{2}$. While this is true, when we write it out in terms of events, we can see that this is not the quantity that we want. See part (b) for more details.

(b) In this part, we want to find $Pr[(B_1 \cap B_2)|B_1]$:

$$\begin{aligned}
Pr[(B_1 \cap B_2)|B_1] &= \frac{Pr[(B_1 \cap B_2) \cap B_1]}{Pr[B_1]} \\
&= \frac{Pr[B_1 \cap B_2]}{Pr[B_1]} \\
&= \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}
\end{aligned} \tag{1}$$

Note the distinction between this part and part (a), and that a common mistake in determining the answer to part (a) is solving part (b) instead.

3. **Bayes Rule - Man Speaks Truth**

(a) A man speaks the truth 3 out of 4 times. He flips a biased coin that comes up Heads $\frac{1}{3}$ of the time and reports that it is Heads. What is the probability it is Heads?

(b) A man speaks the truth 3 out of 4 times. He rolls a fair 6-sided dice and reports it comes up 6. What is the probability it is really 6?

Solution:

(a) Let E denote the event the man reports heads, S_1 be the event that the coin comes up heads, and S_2 be the event that the coin comes up tails.
We have: $P(E|S_1) = \frac{3}{4}, P(E|S_2) = \frac{1}{4}, P(S_1) = \frac{1}{3}, P(S_2) = \frac{2}{3}$.
We want to compute $P(S_1|E)$, and let's do so by applying Bayes Rule.

$P(S_1|E) = \frac{P(S_1 E)}{P(E)} = \frac{P(E|S_1)P(S_1)}{P(E|S_1)P(S_1)+P(E|S_2)P(S_2)} = \frac{3/4 \cdot 1/3}{3/4 \cdot 1/3 + 1/4 \cdot 2/3} = \frac{3}{5}$.

(b) Let E denote the event the man reports 6, S_1 be the event that the dice comes up 6, and S_2 be the event that the dice comes up not 6.
We have: $P(E|S_1) = \frac{3}{4}, P(E|S_2) = \frac{1}{4}, P(S_1) = \frac{1}{6}, P(S_2) = \frac{5}{6}$.
We want to compute $P(S_1|E)$, and let's do so by applying Bayes Rule.

$P(S_1|E) = \frac{P(S_1 E)}{P(E)} = \frac{P(E|S_1)P(S_1)}{P(E|S_1)P(S_1)+P(E|S_2)P(S_2)} = \frac{3/4 \cdot 1/6}{3/4 \cdot 1/6 + 1/4 \cdot 5/6} = \frac{3}{8}$.

4. **Disease diagnosis**

You have a high fever and go to the doctor to identify the cause. 1% of the people have H1N1, 10% of the people have the flu, and 89% have neither. Assume that no person has both. Suppose that 100% of the H1N1 people have a high fever, 30% of the flu people have a high fever, and 2% of the people who have neither, have a high fever. Is it more likely that you have H1N1, the flu, or neither?

Solution: Let A be the event that the patient has H1N1, B be the event that the patient has Flu, and C be the event that the patient has neither. The event of having a fever is D. We want to compare $Pr(A|D)$, $Pr(B|D)$, and $Pr(C|D)$. We find each value using Bayes' rule.

$$\begin{aligned} Pr(A|D) &= \frac{Pr(D|A)Pr(A)}{Pr(D|A)Pr(A)+Pr(D|B)Pr(B)+Pr(D|C)Pr(C)} \\ &= \frac{1 \times 0.01}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} \\ &= 0.173 \end{aligned} \tag{2}$$

$$\begin{aligned} Pr(B|D) &= \frac{Pr(D|B)Pr(B)}{Pr(D|A)Pr(A)+Pr(D|B)Pr(B)+Pr(D|C)Pr(C)} \\ &= \frac{0.1 \times 0.3}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} \\ &= 0.519 \end{aligned} \tag{3}$$

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\begin{aligned}
Pr(C|D) &= \frac{Pr(D|C)Pr(C)}{Pr(D|A)Pr(A)+Pr(D|B)Pr(B)+Pr(D|C)Pr(C)} \\
&= \frac{0.89\times 0.02}{1\times 0.01+0.1\times 0.3+0.89\times 0.02} \\
&= 0.308
\end{aligned} \tag{4}
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So flu is the most likely.