CS 70 Discrete Mathematics and Probability Theory
Fall 2016 Seshia and Walrand **Discussion 9B**

1. **Rain and Wind**

   The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in EECS70, you are curious to play around with these numbers. Find the probability that:

   (a) A given day is both windy and rainy.

   (b) A given day is rainy.

   (c) For a given pair of days, exactly one of the two days is rainy.

   (d) A given day that is non-rainy is also non-windy.

   Solution:

   (a) Let R be the event that it rains on a given day and W be the event that a given day is windy. We are given $P(R|W) = 0.3$, $P(R|W^C) = 0.8$ and $P(W) = 0.2$. Then probability that a given day is both rainy and windy is $P(R \cap W) = P(R|W)P(W) = 0.3 \times 0.2 = 0.06$

   (b) Probability that it rains on a given day is $P(R) = P(R|W)P(W) + P(R|W^C)P(W^C) = 0.3 \times 0.2 + 0.8 \times 0.8 = 0.7$

   (c) Let R_1 and R_2 be the events that it rained on day 1 and day 2 respectively. Since the days are independent, $P(R_1) = P(R_2) = P(R)$. The required probability is $P(R_1)P(R_2^C) + P(R_1^C)P(R_2) = 2 * 0.7 * 0.3 = 0.42$

   (d) Probability that a non-rainy day is non-windy is $P(W^C|R^C) = \frac{P(W^C \cap R^C)}{P(R^C)} = \frac{P(R^C|W^C)(W^C)}{P(R^C)} = \frac{0.2 \times 0.8}{0.3} = \frac{8}{15}$

2. **Balls and Bins**

   Throw n balls into n bins.

   (a) What is the probability that the first bin is empty?

   (b) What is the probability that the first k bins are empty?

   (c) What is the probability that the second bin is empty given that the first one is empty?

   (d) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

   (e) Are the events that "the first bin is empty" and "the second bin is empty" independent?

Solution:

(a) $\left(\frac{n-1}{n}\right)^n$

(b) $\left(\frac{n-k}{n}\right)^n$

(c) Using probability rules:

$$\begin{aligned}\Pr[\text{2nd bin empty} \mid \text{1st bin empty}] &= \frac{\Pr[\text{2nd bin empty} \cap \text{1st bin empty}]}{\Pr[\text{1st bin empty}]} \\ &= \frac{\left(\frac{n-2}{n}\right)^n}{\left(\frac{n-1}{n}\right)^n} \\ &= \left(\frac{n-2}{n-1}\right)^n\end{aligned}$$

Alternate solution:

We know bin 1 is empty, so each ball that we throw can land in one of the remaining $n-1$ bins. We want the probability that bin 2 is empty, which means that each ball cannot land in bin 2 either, leaving $n-2$ bins. Thus for each ball, the probability that bin 2 is empty given that bin 1 is empty is $\frac{n-2}{n-1}$. For n total balls, this probability is $\left(\frac{n-2}{n-1}\right)^n$.

(d) They are dependent. Knowing the latter means the former happens with probability 1.

(e) In part (c) we calculated the probability that the second bin is empty given that the first bin is empty: $\left(\frac{n-2}{n-1}\right)^n$. The probability that the second bin is empty (without any given information) is $\left(\frac{n-1}{n}\right)^n$. Since these probabilities are not equal, the events are dependent.

3. **Birthdays**

Suppose you record the birthdays of a large group of people, one at a time until you have found a match, i.e., a birthday that has already been recorded. (Assume there are 365 days in a year.)

(a) What is the probability that after the first 3 people's birthdays are recorded, no match has occurred (i.e. each person has a unique birthday)?

(b) What is the probability that the first 3 people all share the same birthday?

(c) What is the probability that it takes more than 20 people for a match to occur?

(d) What is the probability that it takes exactly 20 people for a match to occur?

(e) Suppose instead that you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur?

Solution:

(a) $\frac{364}{365} * \frac{363}{365}$

(b) $(\frac{1}{365})^2$

(c) Pr[it takes more than 20 people] $=$ Pr[20 people don't have the same birthday] $= \frac{\frac{365!}{(365-20)!}}{365^{20}} =$ $\boxed{\frac{\frac{365!}{345!}}{365^{20}}} \approx .589$

Another explanation that does not use counting:
Let b_i be the birthday of the i-th person.

$$\begin{aligned}
&\Pr[\text{it takes more than 20 people}]\\
=&\Pr[b_{20} \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 19] \times \Pr[b_i\text{'s are all different } \forall 1 \leq i \leq 19]\\
=&\Pr[b_{20} \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 19]\\
&\quad \times \Pr[b_{19} \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 18]\\
&\quad \times \cdots \times \Pr[b_3 \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 2] \times \Pr[b_2 \neq b_1]\\
=&\frac{365-19}{365} \times \frac{365-18}{365} \times \cdots \times \frac{363}{365} \times \frac{364}{365}\\
\approx&.589
\end{aligned}$$

(d) Pr[it takes exactly 20 people] $=$
Pr[first 19 have different birthdays **and** 20^{th}person shares a birthday with one of the first 19]. How total ways can the birthdays be chosen for 20 people? 365^{20}. How many ways can the birthdays be chosen so the first 19 have different birthdays and the 20^{th} person shares a birthday with the first 19? Well, the first person has 365 choices, the second has 364 choices left, and so on until the nineteenth person has $(365-19+1) = 347$ choices left. Then, the 20^{th} person has 19 choices for his birthday. So in total, there are $365 \cdot 364 \cdot \cdots \cdot 348 \cdot 347 \cdot 19 = \frac{365!}{346!} \cdot 19$ ways of getting what we want. So Pr[it takes exactly 20 people] $= \frac{365 \cdot 364 \cdot \cdots \cdot 348 \cdot 347 \cdot 19}{365^{20}} = \boxed{\frac{\frac{365!}{346!} \cdot 19}{365^{20}}} \approx .032$

Another explanation that does not use counting:
Let b_i be the birthday of the i-th person.

$$\begin{aligned}
&\Pr[\text{it takes exactly 20 people}]\\
=&\Pr[b_{20} \text{ is equal to one of the } b_i\text{'s} \mid b_i\text{'s are all different } \forall 1 \leq i \leq 19]\\
&\quad \times \Pr[b_i\text{'s are all different } \forall 1 \leq i \leq 19]\\
=&\Pr[b_{20} \text{ is equal to one of the } b_i\text{'s} \mid b_i\text{'s are all different } \forall 1 \leq i \leq 19]\\
&\quad \times \Pr[b_{19} \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 18]\\
&\quad \times \cdots \times \Pr[b_3 \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 2] \times \Pr[b_2 \neq b_1]\\
=&\frac{19}{365} \times \frac{365-18}{365} \times \cdots \times \frac{363}{365} \times \frac{364}{365}\\
\approx&.032
\end{aligned}$$

(e) Pr[it takes exactly 20 people] $=$
Pr[first 19 don't have your birthday **and** 20^{th}person has your birthday].

Similar to the last problem, there are 364 choices for the first person's birthday to be different than yours, 364 for the second person, and so on until the nineteenth person has 364 choices. Then, the 20^{th} person has exactly 1 choice to have your birthday. So the total number of ways to get what we want is $364^{19} \cdot 1$. There are 365^{20} possibilities total. So $\Pr[\text{it takes exactly 20 people}] = \boxed{\mathbf{\frac{364^{19}}{365^{20}}}} \approx .0026$

Another explanation that does not use counting:

$$\begin{aligned}
\Pr[\text{it takes exactly 20 people}] =& \Pr[\text{the 1st person does not have the same birthday as yours}] \\
& \times \Pr[\text{the 2nd person does not have the same birthday as yours}] \\
& \times \cdots \\
& \times \Pr[\text{the 19th person does not have the same birthday as yours}] \\
& \times \Pr[\text{the 20th person has the same birthday as yours}] \\
=& \frac{364}{365} \times \frac{364}{365} \times \cdots \frac{364}{365} \times \frac{1}{365} \\
=& \frac{364^{19} \times 1}{365^{20}} \\
\approx& 0.0026
\end{aligned}$$