1. **Locked Out**

   You just rented a large house and the realtor gave you five keys, one for the front door and the other four for each of the four side and back doors of the house. Unfortunately, all keys look identical, so to open the front door, you are forced to try them at random.

   Find the distribution and the expectation of the number of trials you will need to open the front door. (Assume that you can mark a key after you've tried opening the front door with it and it doesn't work.)

2. **Linearity**

   Solve each of the following problems using linearity of expectation. Explain your methods clearly.

   (a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

   (b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears?

   (c) A building has n floors numbered $1,2,\ldots,n$, plus a ground floor G. At the ground floor, m people get on the elevator together, and each gets off at a uniformly random one of the n floors (independently of everybody else). What is the expected number of floors the elevator stops at (not counting the ground floor)?

   (d) A coin with heads probability p is flipped n times. A "run" is a maximal sequence of consecutive flips that are all the same. (Thus, for example, the sequence $HTHHHTTH$ with $n=8$ has five runs.) Show that the expected number of runs is $1+2(n-1)p(1-p)$. Justify your calculation carefully.

3. **Coupon Collection**

   Suppose you take a deck of n cards and repeatedly perform the following step: take the current top card and put it back in the deck at a uniformly random position. (The probability that the card is placed in any of the n possible positions in the deck — including back on top — is $1/n$.) Consider the card that starts off on the bottom of the deck. What is the expected number of steps until this card rises to the top of the deck? (Hint: Let T be the number of steps until the card rises to the top. We have $T = T_n + T_{n-1} + \cdots + T_2$, where the random variable T_i is the number of steps until the bottom card rises from position i to position $i-1$. Thus, for example, T_n is the number of steps until the bottom card rises off the bottom of the deck, and T_2 is the number of steps until the bottom card rises from second position to top position. What is the distribution of T_i?)