1. **Uniform Probability Space**

   Let $\Omega = \{1,2,3,4,5,6\}$ be a uniform probability space. Let also $X(\omega)$ and $Y(\omega)$, for $\omega \in \Omega$, be the random variables defined as follows:

   Table 1: The random variables X and Y.

ω	1	2	3	4	5	6
$X(\omega)$	0	0	1	1	2	2
$Y(\omega)$	0	2	3	5	2	0

   (a) Calculate $V = L[Y|X]$;
   (b) Calculate $W = E[Y|X]$;
   (c) Calculate $E[(Y-V)^2]$;
   (d) Calculate $E[(Y-W)^2]$.

   [*Hint:* Recall that $L[Y|X]$ and $E[Y|X]$ are functions of X and that you need to specify their value as a function of X.]

   Solution:

   (a) We find $E[X] = 1, E[Y] = 2, E[XY] = 2$, so that $cov(X,Y) = 0$ and $L[Y|X] = E[Y] = 2$.
   (b) We see that $E[Y|X=0] = 1, E[Y|X=1] = 4, E[Y|X=2] = 1$.
   (c) $E[(Y-V)^2] = E[(Y-2)^2] = (4+0+1+9+0+4)/6 = 3$.
   (d) $E[(Y-W)^2] = (1+1+1+1+1+1)/6 = 1$.

2. **Number of Ones**

   In this problem, we will revisit dice-rolling, except with conditional expectation.

   (a) If we roll a die until we see a 6, how many ones should we expect to see?
   (b) If we roll a die until we see a number greater than 3, how many ones should we expect to see?

   Solution:

(a) Let Y be the number of ones we see. Let X be the number of rolls we take until we get a 6. We are, then, solving for $E[Y|X]$.
Let us first compute $E[Y|X]$. We know that since in each of our $k-1$ rolls before the kth, we necessarily roll a number in $\{1, 2, 3, 4, 5\}$. Thus, we have a $\frac{1}{5}$ chance of getting a one, meaning $E[Y|X=k] = \frac{1}{5}(k-1)$, so $E[Y|X] = \frac{1}{5}(X-1)$.
If this is confusing, write Y as a sum of indicator variables.

$$Y = Y_1 + Y_2 + ...Y_k$$

where Y_i is 1 if we see a one on the ith roll. This means

$$E[Y|X=k] = E[Y_1|X=k] + E[Y_2|X=k] + ...E[Y_k|X=k]$$

We know for a fact that on the kth roll, we roll a 6, thus $E[Y_k] = 0$. Thus, we actually consider

$$\begin{aligned} E[Y_1|X=k] + E[Y_2|X=k] + ...E[Y_{k-1}|X=k] &= (k-1)E[Y_1|X=k] \\ &= (k-1)Pr[Y_1|X=k] \\ &= (k-1)\frac{1}{5} \end{aligned}$$

Using the Law of Total Expectation, we know that

$$\begin{aligned} E[E[Y|X]] = E[Y] &= E[\frac{1}{5}(X-1)] \\ &= \frac{1}{5}E[X-1] \\ &= \frac{1}{5}(E[X]-1) \end{aligned}$$

Since, X $Geom(\frac{1}{6})$, the expected number of rolls until we roll a 6 is $E[X] = 6$.

$$\frac{1}{5}(E[X]-1) = \frac{1}{5}(6-1) = 1$$

(b) The first change from the previous part is the probability of rolling a 1, given we have made k rolls to get to our first roll that satisfies the condition. This makes

$$\begin{aligned} E[E[Y|X]] = E[Y] &= E[\frac{1}{3}(X-1)] \\ &= \frac{1}{3}(E[X]-1) \end{aligned}$$

Since X $Geom(\frac{1}{2})$, we know that the expected number of rolls until we roll a number greater than 3 is $E[X] = 2$.
This makes $E[Y] = \frac{1}{3}$.

3. **Marbles in a Bag**

We have r red marbles, b blue marbles and g green marbles to the same bag. If we sample balls, with replacement, until we get 3 red marbles (not necessarily consecutively), how many blue marbles should we expect to see?

Solution: Let Y be the number of blue marbles we see. Let X be the samples we take until we get 3 red marbles. We are, then, solving for $E[Y|X]$.

Let us first compute $E[Y|X]$. Let Y_i be 1 if we see a blue marble on the ith roll and $Y = \sum_i^k Y_i$. This means

$$\begin{aligned} E[Y|X=k] &= E[\sum_i^k Y_i|X=k] \\ &= \sum_i^k E[Y_i|X=k] \end{aligned}$$

However, We also three Y_i have $E[Y_i] = 0$, since there are necessarily 3. This means the other $k-3$ marbles are necessarily blue or green.

$$\begin{aligned} \sum_i^k E[Y_i|X=k] &= \sum_{i \neq a,b,c}^k Pr[Y_i|X=k] \\ &= \sum_{i \neq a,b,c}^k \frac{b}{b+g} \\ &= (k-3)\frac{b}{b+g} \end{aligned}$$

This means $E[Y|X] = (X-3)\frac{b}{b+g}$.

Using the Law of Total Expectation, we know that

$$\begin{aligned} E[E[Y|X]] = E[Y] &= E[\frac{b}{b+g}(X-3)] \\ &= \frac{b}{b+g}E[X-3] \\ &= \frac{b}{b+g}(E[X]-3) \end{aligned}$$

We know that $E[X] = 3\frac{r+g+b}{r}$.

$$E[Y] = \frac{b}{b+g}(3\frac{r+g+b}{r} - 3)$$