1. **Period of States**

   Calculate explicitly $d(0)$, $d(1)$, $d(2)$, and $d(3)$, defined as

   $$d(i) = \gcd\{n > 0 \mid \Pr[X_n = i \mid X_0 = i] > 0\}$$

   for the Markov chain of Figure **??**. That is, for each state i, identify the set

   $$\{n > 0 \mid \Pr[X_n = i \mid X_0 = i] > 0\}$$

   and finds its g.c.d.

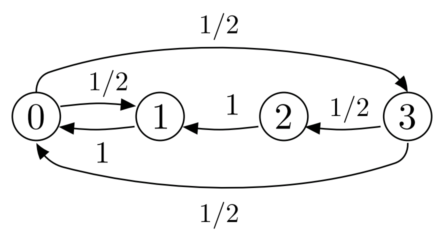

Figure 1: A Markov chain diagram.

2. **Limiting Distribution**

   This problem invites you to test your understanding of the limiting distribution of a Markov chain.

   (a) Construct a Markov chain that is not irreducible but that has a unique distribution and is such that its distribution converges to that unique invariant distribution, for any initial distribution.

   (b) Show a Markov chain whose distribution converges to a limit that depends on the initial distribution.

3. **Skipping Stones**

   We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be $\mathscr{K} = \{1,2,3,4,5\}$. State 3 represents the target, while states 4 and 5 indicate that you have overshot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of $\{3\}$ before $\{4,5\}$.

4. **Allen's Umbrellas**

   Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

   We will model this as a Markov chain. Let $\mathscr{K} = \{0, 1, 2\}$ be the set of states, where the state i represents the number of umbrellas in his current location. Write down the transition matrix, determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

5. **High and Low States**

   Suppose that we have n "high" states $H_1, \ldots, H_n$ and n "low" states $L_1, \ldots, L_n$. The high state H_k has a probability p of transitioning to L_k, and a probability $1 - p$ of staying at H_k. The low state L_k has a probability q of transitioning to the next high state H_{k+1} (wrapping around, so L_n can transition to H_1), and a probability $1 - q$ of staying at the same location. Is the Markov chain aperiodic? What is the limiting distribution?