1. **Period of States**

Calculate explicitly $d(0)$, $d(1)$, $d(2)$, and $d(3)$, defined as

$$d(i) = \gcd\{n > 0 \mid \Pr[X_n = i \mid X_0 = i] > 0\}$$

for the Markov chain of Figure 1. That is, for each state $i$, identify the set

$$\{n > 0 \mid \Pr[X_n = i \mid X_0 = i] > 0\}$$

and finds its g.c.d.

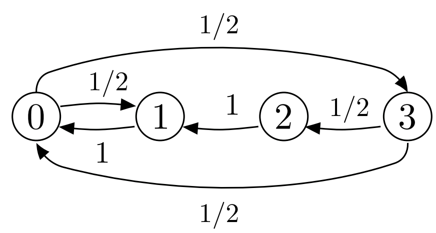


Figure 1: A Markov chain diagram.

**Solution:**

(a) We see that
$$d(0) = \gcd\{2, 4, 6, \ldots\} = 2.$$

(b) Similarly
$$d(1) = \gcd\{2, 4, 6, \ldots\} = 2.$$

(c) One has
$$d(2) = \gcd\{4, 6, 8, \ldots\} = 2.$$

(d) Finally,
$$d(3) = \gcd\{2, 4, \ldots\} = 2.$$

Since the Markov chain is irreducible, all the states have the same period.

2. **Limiting Distribution**

This problem invites you to test your understanding of the limiting distribution of a Markov chain.

(a) Construct a Markov chain that is not irreducible but that has a unique distribution and is such that its distribution converges to that unique invariant distribution, for any initial distribution.

(b) Show a Markov chain whose distribution converges to a limit that depends on the initial distribution.

**Solution:**

(a) We saw that if a Markov chain is irreducible and aperiodic, it has a unique invariant distribution and its distribution converges to that invariant distribution. The point of this problem is to stress the fact that a Markov chain that is not irreducible may also have these properties.

A simple example is a Markov chain with two states 0 and 1 such that $P(0,1) = a > 0$ and $P(1,1) = 1$. This Markov chain is not irreducible since it cannot go from 1 to 0. It has a unique invariant distribution which is $\pi = [0,1]$. Also, it is clear that $\pi_n \to \pi$ as $n \to \infty$, for all choices of $\pi_0$.

(b) The simplest example is a Markov chain that does not move, i.e. whose transition probability matrix is the identity matrix.

3. **Skipping Stones**

We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be $\mathscr{K} = \{1,2,3,4,5\}$. State 3 represents the target, while states 4 and 5 indicate that you have overshot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of $\{3\}$ before $\{4,5\}$.

**Solution:**

Let $\alpha(i)$ denote the probability of reaching the target before overshooting, starting at state $i$. Then

$$\begin{aligned}
\alpha(5) &= 0 \\
\alpha(4) &= 0 \\
\alpha(3) &= 1 \\
\alpha(2) &= \frac{1}{3}\alpha(3) + \frac{1}{3}\alpha(4) + \frac{1}{3}\alpha(5) = \frac{1}{3} \\
\alpha(1) &= \frac{1}{3}\alpha(2) + \frac{1}{3}\alpha(3) + \frac{1}{3}\alpha(4) = \frac{1}{9} + \frac{1}{3}
\end{aligned}$$

Therefore, $\alpha(1) = 1/9 + 1/3 = 4/9$.

4. **Allen's Umbrellas**

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is $p$.

We will model this as a Markov chain. Let $\mathscr{K} = \{0, 1, 2\}$ be the set of states, where the state $i$ represents the number of umbrellas in his current location. Write down the transition matrix, determine if the distribution of $X_n$ converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

**Solution:**

Suppose Allen is in state 0. Then, Allen has no umbrellas to bring, so with probability 1 Allen arrives at a location with 2 umbrellas. That is,

$$\Pr[X_{n+1} = 2 \mid X_n = 0] = 1$$

Suppose Allen is in state 1. With probability $p$, it rains and Allen brings the umbrella, arriving at state 2. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 1.

$$\Pr[X_{n+1} = 2 \mid X_n = 1] = p, \Pr[X_{n+1} = 1 \mid X_n = 1] = 1-p$$

Suppose Allen is in state 2. With probability $p$, it rains and Allen brings the umbrella, arriving at state 1. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 0.

$$\Pr[X_{n+1} = 1 \mid X_n = 2] = p, \Pr[X_{n+1} = 0 \mid X_n = 2] = 1-p$$

We summarize this with the transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$

Observe that the transition matrix is irreducible and aperiodic, so it converges to its invariant distribution. To solve for the distribution, we set $\pi P = \pi$, or $\pi(P-I) = 0$. This yields the balance equations

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & p \\ 1-p & p & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

As usual, one of the equations is redundant. We replace the last column by the normalization condition $\pi(0) + \pi(1) + \pi(2) = 1$.

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & 1 \\ 1-p & p & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Now solve for the distribution:

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix} = \frac{1}{3-p}\begin{bmatrix}1-p & 1 & 1\end{bmatrix}$$

The invariant distribution also tells us the long-term fraction of time that Allen spends in each state. We can see that Allen spends a fraction $(1-p)/(3-p)$ of his time with no umbrella in his location, so the long-term fraction of time in which he walks through rain is $p(1-p)/(3-p)$.

5. **High and Low States**

Suppose that we have $n$ "high" states $H_1,\dots,H_n$ and $n$ "low" states $L_1,\dots,L_n$. The high state $H_k$ has a probability $p$ of transitioning to $L_k$, and a probability $1-p$ of staying at $H_k$. The low state $L_k$ has a probability $q$ of transitioning to the next high state $H_{k+1}$ (wrapping around, so $L_n$ can transition to $H_1$), and a probability $1-q$ of staying at the same location. Is the Markov chain aperiodic? What is the limiting distribution?

**Solution:**

Yes, the Markov chain is aperiodic because each state can transition to itself.

To compute the limiting distribution, use symmetry. Notice that the "high" states are symmetric, as are the "low" states. In other words, we expect $\pi(H_1) = \cdots = \pi(H_n)$ and $\pi(L_1) = \cdots = \pi(L_n)$. Now, group all of the "high" states into one state $H$ and all of the "low" states into one state $L$: then, we reduce the problem to a two-state Markov chain. By inspection, we can see that the limiting distribution for this two-state Markov chain is

$$\begin{bmatrix}\pi(H) & \pi(L)\end{bmatrix} = \frac{1}{p+q}\begin{bmatrix}q & p\end{bmatrix}$$

Now, by symmetry, the probability $\pi(H)$ should be evenly distributed among the $n$ "high" states, and similarly for the "low" states.

$$\pi(H_i) = \frac{q}{n(p+q)}$$
$$\pi(L_i) = \frac{p}{n(p+q)}$$