1. **Markov Chains: Prove/Disprove**

   (a) There exists an irreducible, finite Markov Chain for which there exist initial distributions that converge to different distributions.

   (b) There exists an irreducible, aperiodic, finite Markov Chain for which $P(X_{n+1} = j|X_n = i) = 1$ or 0 for all i, j.

   (c) There exists an irreducible, non-aperiodic Markov Chain for which $P(X_{n+1} = j|X_n = i) \neq 1$ for all i, j.

   (d) For an irreducible, non-aperiodic Markov Chain, any initial distribution not equal to the invariant distribution does not converge to any distribution.

Solution:

(a) False. Every finite irreducible Markov Chain has a unique stationary distribution. If it's possible for the Markov chain to converge to two different distributions given different starting distributions, it implies there are two stationary distributions. To elaborate further, we know in the long run the fraction of time spent in each state converges to the stationary distribution. So if the distribution converges, the long-run fraction of time will be whatever distribution it converges to, which we see must be the stationary distribution.

(b) True, you can have one state pointing to itself. However for number of states >1 it is false. Consider the initial distribution of having a probability of 1 of being in an arbitrary state. After a transition, the resulting distribution must be a probability 1 of being in a different state (if it were the same state, this would immediately imply that the Markov Chain is reducible). Further transitions have the same effect. Therefore this initial distribution does not converge. Therefore this Markov Chain cannot be aperiodic and irreducible (since it would converge in that case).

(c) True. Consider the states $\{0,1,2,3\}$. Set $P(i, j) = \frac{1}{2}$ if $i \equiv j \pm 1 \mod 4$ and 0 otherwise. In other words, the Markov Chain is a square with each side replaced with two links pointing in opposite directions with probabilities of $\frac{1}{2}$. Consider the period of state 0. Any path from 0 back to itself, such as $0-1-2-1-0$, alternates in parity of each consecutive state since each state only points to the state above or below it mod 4. Therefore state 0 has period 2. Therefore this Markov Chain is not aperiodic (and all states have period 2).

(d) False. Take the initial distribution $[0.25, 0.30, 0.25, 0.20]$ for the above Markov Chain. After one transition it goes to the invariant distribution, $[0.25, 0.25, 0.25, 0.25]$.

2. **Pokemon Craze**

You and your friend are both trying to catch a Dratini. Unfortunately, you each can only attempt to catch one Dratini per day. Once you or your friend catch a Dratini, that person stops while the other person continues to try to catch a Dratini if they haven't already caught one. The probability an attempt at catching a Dratini is successful is p. What is the expected number of days this process takes, if you two both try to catch a Dratini every day? Solve this using a Markov Chain with three states. (What is this in terms of two Geometric random variables?) (Also, consider how this problem would change if instead of stopping, the first person kept on trying to catch a Dratini in case he could donate it to the other person).

Solution: The three states are $\{S, D, DD\}$. S represents neither person having a Dratini, D represents one person having a Dratini, and DD is when both have a Dratini. We can write hitting time equations:

$$E[S] = 1 + p^2 E[DD] + (1-p)^2 E[S] + 2p(1-p)E[D]$$
$$E[D] = 1 + (1-p)E[D] + p(E[DD])$$
$$E[DD] = 0$$

$p^2, (1-p)^2, 2p(1-p)$ respectively are the probabilites of both trainers catching a Dratini at the same time, neither catching a Dratini, and one of them catching a Dratini. In the state D, we're only concerned with the remaning person without a Dratini so the probabilities simplify to p and $1-p$. $E[DD] = 0$ because it's the success state. Solving the system we get $E[S] = \frac{2(1-p)+1}{1-(1-p)^2} = \frac{2}{p} - \frac{1}{1-(1-p)^2}$.
Notice this is the max of two Geometric random variables. We could solve this problem by finding $Pr[Max > k]$ and using the tail sum to find the expectation, which would give the right expression (which is equal to the left but looks different).
If instead the first person who caught a Dratini decided to help the other person out, we would replace the equation for $E[D]$ with $E[D] = 1 + (1-p)^2 E[D] + (2p - p^2)E[DD]$.

3. **Continuous Intro**

   (a) Is $f(x) = 2x$ from $0 \le x \le 1$, 0 otherwise a valid pdf? Why or why not? Is it a valid cdf? Why or why not?

   (b) Calculate $E[X]$ and $Var(X)$ for X with pdf $f(x) = 1/l$ from $0 \le x \le l$, 0 otherwise.

   (c) Suppose X and Y are independent and have pdfs $f(x) = 2x$ from $0 \le x \le 1$, 0 otherwise and $f(y) = 1$ from $0 \le x \le 1$. What is their joint distribution?

   (d) Calculate $E[XY]$ for the above X and Y.

Solution:

   (a) Yes; it is nonnegative and integrates to 1. No; cdf should go to 1 as x goes to infinity and be nondecreasing.

   (b) $E[X] = \int_{x=0}^{l} x \cdot (1/l)\,dx = l/2$. $E[X^2] = \int_{x=0}^{l} x^2 \cdot (1/l)\,dx = l^2/3$. $Var(X) = l^2/3 - l^2/4 = l^2/12$.

(c) Note that due to independence, $f(x,y)dxdy = Pr(X \in [x,x+dx], Y = [y,y+dy]) = Pr(X \in [x+dx])Pr(Y \in [y+dy]) \approx f(x)f(y)dxdy$ so their joint distribution is $f(x,y) = 2x$, for the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

(d) $E[XY] = \int_{x=0}^{1}\int_{y=0}^{1} xy(2x)dydx = \int_{x=0}^{1} x^2 dx = 1/3$.

4. **Uniform Distribution**You have two spinners, each having a circumference of 10, with values in the range $[0,10)$. If you spin both (independently) and let X be the position of the first spinner and Y be the position of the second spinner, what is the probability that $X \geq 5$, given that $Y \geq X$?

   Solution: First we write down what we want and expand out the conditioning:

$$\Pr[X \geq 5 \,|\, Y \geq X] = \frac{\Pr[Y \geq X \cap X \geq 5]}{\Pr[Y \geq X]}.$$

   $\Pr[Y \geq X] = \frac{1}{2}$ by symmetry. To find $\Pr[Y \geq X \cap X \geq 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law ($\Pr[A] = \frac{\text{area of } A}{\text{area of } \Omega}$). We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

$$\Pr[Y \geq X \cap X \geq 5] = \frac{\frac{5 \cdot 5}{2}}{10 \cdot 10} = \frac{1}{8}.$$

   So $\Pr[X \geq 5 \,|\, Y \geq X] = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$.