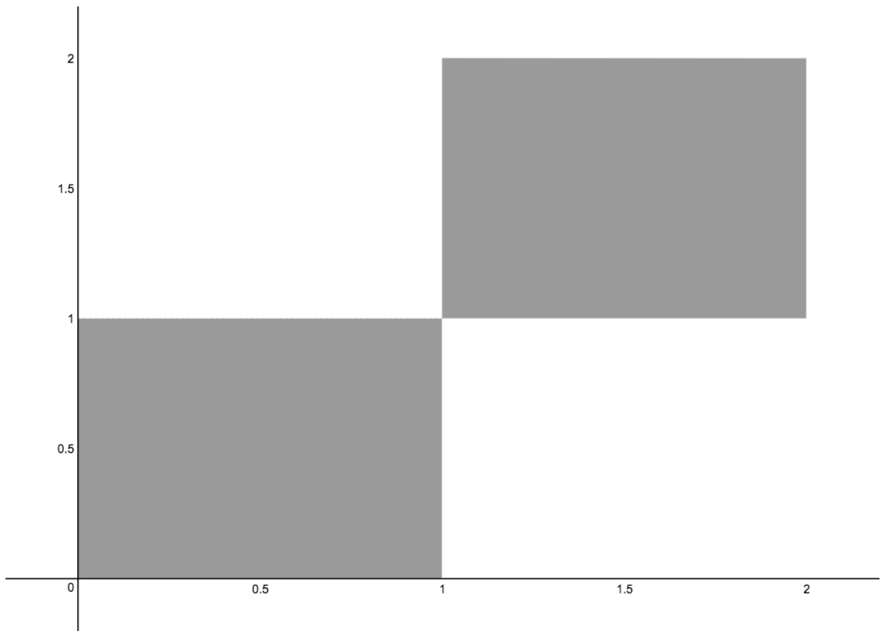
1. **Continuous LLSE**

Suppose that X and Y are uniformly distributed on the following figure:



That is, X and Y have the joint distribution

$$f_{X,Y}(x,y) = \begin{cases} 1/2, & 0 \le x \le 1, 0 \le y \le 1 \\ 1/2, & 1 \le x \le 2, 1 \le y \le 2 \end{cases}$$

(a) Do you expect X and Y to be positively correlated, negatively correlated, or neither?

(b) Compute the marginal distribution of X.

(c) Compute $L[Y \mid X]$.

(d) What is $E[Y \mid X]$?

Solution:

(a) Positively correlated, because high values of Y correspond to high values of X.

(b) Intuitively, if we slice the joint distribution at any $x \in [0,2]$, then the probability is the same, so we should expect X to be uniformly distributed on $[0,2]$. We verify this by explicit computation:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = 1\{0 \le x \le 1\} \int_0^1 \frac{1}{2}\,\mathrm{d}y + 1\{1 \le x \le 2\} \int_1^2 \frac{1}{2}\,\mathrm{d}y$$
$$= \frac{1}{2} 1\{0 \le x \le 2\}$$

(c) $E[X] = E[Y] = 1$ by symmetry. Since X is uniform on $[0,2]$, $\text{var}(X) = 4 \cdot 1/12 = 1/3$ (since the variance of a $\text{Unif}[0,1]$ random variable is $1/12$). We compute the covariance:

$$E[XY] = \int_{-\infty}^{-\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^1 xy \cdot \frac{1}{2}\,\mathrm{d}x\,\mathrm{d}y + \int_1^2 \int_1^2 xy \cdot \frac{1}{2}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \frac{1}{2}\left(\int_0^1 x\,\mathrm{d}x \int_0^1 y\,\mathrm{d}y + \int_1^2 x\,\mathrm{d}y \int_1^2 y\,\mathrm{d}y\right) = \frac{1}{2}\left(\frac{1}{4} + \frac{9}{4}\right) = \frac{5}{4}$$

So $\text{cov}(X,Y) = 5/4 - 1 \cdot 1 = 1/4$. The LLSE is

$$L[Y \mid X] - 1 = \frac{1/4}{1/3}(X - 1)$$
$$L[Y \mid X] = \frac{3}{4}X + \frac{1}{4}$$

(d) The easiest way to solve this is to look at the picture of the joint density, and draw horizontal lines through middles of each of the two squares. Intuitively, $E[Y \mid X]$ means "for each slice of $X = x$, what is the best guess of Y"? Slightly more formally, one can argue that conditioned on $X = x$ for $0 < x < 1$, $Y \sim U[0,1]$, so $E[Y \mid X = x] = 1/2$ in this region. Conditioned on $X = x$ for $1 < x < 2$, $Y \sim U[1,2]$, so $E[Y \mid X = x] = 3/2$ in this region.

$$E[Y \mid X = x] = \begin{cases} 1/2, & 0 \le x \le 1 \\ 3/2, & 1 \le x \le 2 \end{cases}$$

2. **Conditioning on Exponentials**

Let X_i be i.i.d. $\text{Expo}(\lambda)$ random variables.

(a) Compute $E[Y \mid Z]$, where $Y = \max\{X_1, X_2\}$ and $Z = \min\{X_1, X_2\}$.

(b) Compute $E[X_1 + X_2 \mid Z]$. (*Hint*: Use part (a).)

(c) Use part (b) to compute $E[Z]$.

(d) Compute $E[X_1 + X_2 \mid X_1 + X_2 + X_3]$.

Solution:

(a) View the exponential distributions as light bulbs burning out. Conditioned on Z, we are looking for the expected time until the next light bulb burns out, which by the memoryless property is $1/\lambda$. Therefore, $E[Y \mid Z] = Z + 1/\lambda$.

(b) Observe that $X_1 + X_2 = \min\{X_1, X_2\} + \max\{X_1, X_2\}$.

$$E[X_1 + X_2 \mid Z] = E[\min\{X_1, X_2\} \mid Z] + E[\max\{X_1, X_2\} \mid Z] = Z + Z + 1/\lambda = 2Z + 1/\lambda$$

(c) Take expectations of both sides and apply iterated expectation.

$$E[X_1 + X_2] = E[E[X_1 + X_2 \mid Z]] = 2E[Z] + \frac{1}{\lambda}$$

Note that $E[X_1 + X_2] = 2/\lambda$ by linearity, so we find that $E[Z] = 1/(2\lambda)$. (In fact, we can prove a stronger result: the minimum of two independent exponentials with parameter λ is exponential with parameter 2λ.)

(d) Given $X_1 + X_2 + X_3$, the i.i.d. nature of the exponentials tells us that we should expect $E[X_1 + X_2 \mid X_1 + X_2 + X_3] = (2/3)(X_1 + X_2 + X_3)$.

3. **Erlang Distribution**

In lecture, we proved the following **Fact**: if the lifetimes of light bulbs are i.i.d. $\text{Expo}(1)$ and we replace the light bulb as soon as one dies out, then the number of light bulbs we replace by time t follows the Poisson distribution with mean t. Using this fact, find the density of the sum of two i.i.d. $\text{Expo}(1)$ random variables.

Solution:

First, we compute the probability that a light bulb will die in the first $\mathrm{d}t$ seconds, where $\mathrm{d}t$ is an infinitesimal amount of time. Let $T \sim \text{Expo}(1)$.

$$\Pr[T \in (0, \mathrm{d}t)] \approx f_T(0)\,\mathrm{d}t = \lambda\,\mathrm{d}t$$

By the memoryless property, this is the same as the probability that the light bulb will die in *any* interval of $\mathrm{d}t$ seconds, conditioned on the fact that the light bulb has survived until that point in time. We compute now $\Pr[T \in (t, t + \mathrm{d}t)]$, the probability that the lifetime of two light bulbs is within $(t, t + \mathrm{d}t)$. In order for this to happen, we must have the second light bulb die in the interval $(t, t + \mathrm{d}t)$, and the probability of this, we have just argued, is $1\,\mathrm{d}t$. What about the first light bulb? We must have replaced exactly one light bulb in the first t seconds, which by the **Fact**, is the probability that a Poisson distribution with mean t takes on the value 1. Putting this together,

$$\Pr[T \in (t, t + \mathrm{d}t)] = 1\,\mathrm{d}t \cdot \frac{t \cdot e^{-t}}{1!} = te^{-t}\,\mathrm{d}t$$

Therefore, we identify the density as

$$f_{T_1+T_2}(t) = te^{-t}, \qquad t > 0$$

This is a special case of the **Erlang** distribution, which is the density of the sum of k i.i.d. $\text{Expo}(\lambda)$ distributions. The intimate connection between the exponential distribution and the Poisson distribution is part of the study of stochastic processes.