1. **Bayesian Darts**

   You play a game of darts with your friend. You are better than he is, and the distances of your darts to the center of the target are i.i.d. Unif$[0,1]$ whereas his are i.i.d. Unif$[0,2]$. To make the game fair, you agree that you will throw one dart and he will throw two darts. The dart closest to the center wins the game. What is the probability that you will win? *Note*: The distances *from the center of the board* are uniform.

   Solution:

   Let X be the distance of your closest dart to the center and Y that of the closest of your friend's darts. Then, for $x \in [0,1]$ and $y \in [0,2]$,

   $$\Pr[X > x] = (1-x) \quad \text{and} \quad \Pr[Y > y] = \left(1 - \frac{y}{2}\right)^2.$$

   Hence,

   $$f_X(x) = -\frac{\mathrm{d}}{\mathrm{d}x}(1-x) = 1, \qquad x \in [0,1].$$

   Also,

   $$\Pr[Y > X \mid X = x] = \left(1 - \frac{x}{2}\right)^2$$

   Thus,

   $$\begin{aligned}\Pr[Y > X] &= E[(1-X/2)^2] = E[1 - X + X^2/4] \\ &= 1 - 1/2 + 1/12 = 7/12\end{aligned}$$

   since $E[X] = 1/2$ and $E[X^2] = \int_0^1 x^2 \,\mathrm{d}x = 1/3$.

2. **Normal Distribution**

   Recall the following facts about the normal distribution: if $X \sim \mathcal{N}(\mu, \sigma^2)$, then the random variable $Z = (X-\mu)/\sigma$ is standard normal, i.e. $Z \sim \mathcal{N}(0,1)$. There is no closed-form expression for the CDF of the standard normal distribution, so we define $\Phi(z) = \Pr[Z \le z]$. You may express your answers in terms of $\Phi(z)$.

   The average jump of a certain frog is 3 inches. However, because of the wind, the frog does not always go exactly 3 inches. A zoologist tells you that the distance the frog travels is normally distributed with mean 3 and variance $1/4$.

   (a) What is the probability that the frog jumps more than 4 inches?

(b) What is the probability that the distance the frog jumps is between 2 and 4 inches?

Solution:

(a) First, we write down the probability we want to find, then transform the probability in order to work with the standard normal.

$$\Pr[X > 4] = \Pr[X - 3 > 1] = \Pr\left[\frac{X-3}{1/2} > 2\right] = \Pr[Z > 2] = 1 - \Phi(2) \approx 0.0228$$

(b) Since the mean of the jump is 3, and the normal distribution is symmetric, we can rewrite the desired probability as

$$\begin{aligned}\Pr[2 < X < 4] &= 1 - (\Pr[X > 4] + \Pr[X < 2]) \\ &= 1 - 2 \cdot \Pr[X > 4]\end{aligned}$$

We have computed $\Pr[X > 4] = 0.0228$ in part (a), so we can plug this in to get 0.9544.

3. **Chebyshev's Inequality vs. Central Limit Theorem**

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with the following distribution:

$$\Pr[X_i = -1] = 1/12; \quad \Pr[X_i = 1] = 9/12; \quad \Pr[X_i = 2] = 2/12.$$

(a) Calculate the expectations and variances of X_i, $\sum_{i=1}^n X_i$, $\sum_{i=1}^n X_i - n$, and

$$Z_n = \frac{\sum_{i=1}^n X_i - n}{\sqrt{n/2}}$$

(b) Use Chebyshev's Inequality to find an upper bound b for $\Pr[|Z_n| \geq 2]$.

(c) Can you use b to bound $\Pr[Z_n \geq 2]$ and $\Pr[Z_n \leq -2]$?

(d) As $n \to \infty$, what is the distribution of Z_n?

(e) We know that if $Z \sim \mathcal{N}(0,1)$, then $\Pr[|Z| \leq 2] = \Phi(2) - \Phi(-2) \approx 0.9545$. As $n \to \infty$, can you provide approximations for $\Pr[Z_n \geq 2]$ and $\Pr[Z_n \leq -2]$?

Solution:

(a) $E[X_i] = -1/12 + 9/12 + 4/12 = 1$, and

$$\text{var}(X_i) = \frac{1}{12} \cdot 2^2 + \frac{9}{12} \cdot 0^2 + \frac{2}{12} \cdot 1^2 = \frac{1}{2}$$

Using linearity of expectation and variance (since the X_i are independent), we find that $E[\sum_{i=1}^n X_i] = n$ and $\text{var}(\sum_{i=1}^n X_i) = n/2$.

Again, by linearity of expectation, $E[\sum_{i=1}^n X_i - n] = n - n = 0$. Subtracting a constant does not change the variance, so $\text{var}(\sum_{i=1}^n X_i - n) = n/2$, as before.

Using the scaling properties of the expectation and variance, $E[Z_n] = 0/\sqrt{n/2} = 0$ and $\text{var}(Z_n) = (n/2)/(n/2) = 1$.

(b)

$$\Pr[|Z_n| \geq 2] \leq \frac{\text{var}(Z_n)}{2^2} = \frac{1}{4}$$

(c) $1/4$ for both, since $\Pr[Z_n \geq 2] \leq \Pr[|Z_n| \geq 2]$ and $\Pr[Z_n \leq -2] \leq \Pr[|Z_n| \geq 2]$.

(d) By the Central Limit Theorem, we know that $Z_n \to \mathcal{N}(0,1)$, the standard normal distribution.

(e) Since $Z_n \to \mathcal{N}(0,1)$, we can approximate $\Pr[|Z_n| \geq 2] \approx 1 - 0.9545 = 0.0455$. By the symmetry of the normal distribution, $\Pr[Z_n \geq 2] = \Pr[Z_n \leq -2] \approx 0.0455/2 = 0.02275$.

4. **Binomial Concentration**

Here, we will prove that the binomial distribution is *concentrated* about its mean as the number of trials tends to ∞. Suppose we have i.i.d. trials, each with a probability of success $1/2$. Let S_n be the number of successes in the first n trials, and define $Z_n = (S_n - n/2)/(\sqrt{n}/2)$.

(a) What are the mean and variance of Z_n?

(b) What is the distribution of Z_n as $n \to \infty$?

(c) Use the bound $\Pr[Z > z] \leq z^{-1} e^{-z^2/2}$ when Z is normally distributed in order to bound $\Pr[S_n/n > 1/2 + \delta]$.

Solution:

(a) 0 and 1, respectively. We made them so, in order to apply the CLT.

(b) The CLT tells us that $Z_n \to \mathcal{N}(0,1)$.

(c) In order to apply the bound, we must apply it to Z_n.

$$\begin{aligned}\Pr[S_n/n > 1/2 + \delta] &= \Pr[(S_n - n/2)/n > \delta] = \Pr[(S_n - n/2)/(\sqrt{n}/2) > 2\delta\sqrt{n}] \\ &\approx \Pr[Z_n > 2\delta\sqrt{n}] \leq (2\delta\sqrt{n})^{-1} e^{-2\delta^2 n}\end{aligned}$$