CS 70 Discrete Mathematics and Probability Theory
Fall 2016 Seshia and Walrand

HW 8

1 Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of hw party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

2 Problems

1. **Counting, Counting, and More Counting (32 points)**
   The only way to learn counting is to practice, practice, practice, so here is your chance to do so. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

   (a) How many 10-bit strings are there that contain exactly 4 ones?

   (b) How many different 13-card bridge hands are there? (A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.)

   (c) How many different 13-card bridge hands are there that contain no aces?

   (d) How many different 13-card bridge hands are there that contain all four aces?

   (e) How many different 13-card bridge hands are there that contain exactly 6 spades?

   (f) How many 99-bit strings are there that contain more ones than zeros?

   (g) How many different anagrams of FLORIDA are there? (An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.)

   (h) How many different anagrams of ALASKA are there?

   (i) How many different anagrams of ALABAMA are there?

   (j) How many different anagrams of MONTANA are there?

   (k) If we have a standard 52-card deck, how many ways are there to order these 52 cards?

   (l) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

   (m) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

   (n) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

   (o) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

   (p) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

2. **Counting (2/3/5/5 points)**

   (a) How many ways are there to arrange n 1s and k 0s into a sequence?

   (b) How many solutions does

   $x_0 + x_1 + \ldots + x_k = n$

   have, if all xs must be non-negative integers?

(c) How many solutions does

$x_0 + x_1 = n$

have, if all xs must be *strictly positive* integers?

(d) How many solutions does

$x_0 + x_1 + \ldots + x_k = n$

have, if all xs must be *strictly positive* integers?

3. **Fermat's Necklace (2/3/5/5 points)**

Let p be a prime number and let k be a positive integer. We have an endless supply of beads. The beads come in k different colors. All beads of the same color are indistinguishable.

(a) We have a piece of string. As a relaxing study break, we want to make a pretty garland by threading p beads onto the string. How many different ways are there construct such a sequence of p beads of k different colors?

(b) Now let's add a restriction. We want our garland to be exciting and multicolored. Now how many different sequences exist? (Your answer should be a simple function of k and p.)

(c) Now we tie the two ends of the string together, forming a circular necklace which lets us freely rotate the beads around the necklace. We'll consider two necklaces equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors—red (R), green (G), and blue (B)—then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are cyclic shifts of each other.)

How many non-equivalent sequences are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[Hint: What follows if rotating all the beads on a necklace to another position produces an identical looking necklace?]

(d) Use your answer to part (c) to prove Fermat's little theorem. (Recall that Fermat's little theorem says that if p is prime and $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$.)

4. **Maze (5/5 points)**

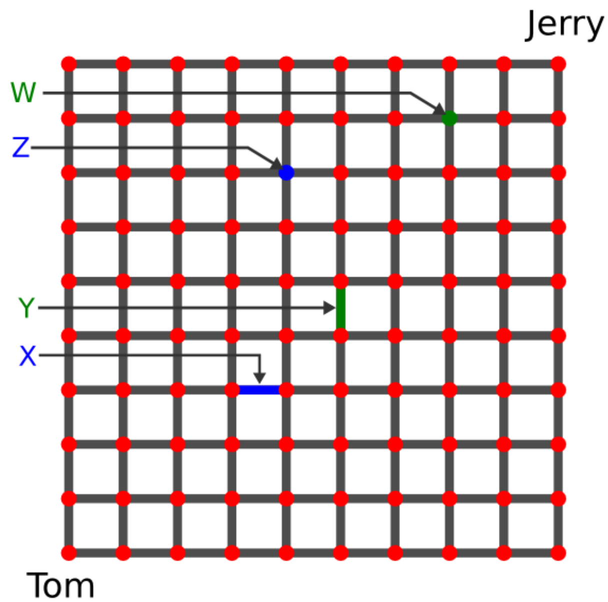
Let's assume that Tom is located at the bottom left corner of the maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.

a) How many such shortest paths exist?

b) How many shortest paths pass through the edge labelled X?

c) **(Optional)** The edge labelled Y? Both the edges X and Y? Neither edge X nor edge Y?

d) **(Optional)** How many shortest paths pass through the vertex labelled Z? The vertex labelled W? Both the vertices Z and W? Neither vertex Z nor vertex W?



5. **Story Problems (5/5/5/5 points)**

   Prove the following identities by combinatorial argument:

   (a)
   $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

   (b)
   $$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

   (c) (Hint: Consider how many ways there are to pick groups of people ("teams") and then a representative ("team leaders").)
   $$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

   (d) (Hint: consider a generalization of the previous part.)
   $$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$$

6. **Sample Space and Events (1/1/1/1/1/1/2/2 points)**
   Consider the sample space Ω of all outcomes from flipping a coin 3 times.

   (a) List all the outcomes in Ω. How many are there?

   (b) Let A be the event that the first flip is a heads. List all the outcomes in A. How many are there?

   (c) Let B be the event that the third flip is a heads. List all the outcomes in B. How many are there?

(d) Let C be the event that the first and third flip are heads. List all outcomes in C. How many are there?

(e) Let D be the event that the first or the third flip is heads. List all outcomes in D. How many are there?

(f) Are the events A and B disjoint? Express C in terms of A and B. Express D in terms of A and B.

(g) Suppose now the coin is flipped $n \geq 3$ times instead of 3 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

(h) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work. (Hint: the answer is NOT 1/2).

7. **To Be Fair (10 points)**

Suppose you have a biased coin with $P(heads) \neq 0.5$. How could you use this coin to simulate a fair coin? (Hint: Think about pairs of tosses.)