CS 70   Discrete Mathematics and Probability Theory
Fall 2016   Seshia and Walrand   HW 10

1   Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of hw party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

2 Problems

1. **Pokemon Again**
   Nintendo comes out with a set of reprinted original Pokemon cards, one for each of the original 150 Pokemon. Naturally nothing will do, but that you collect them all. Suppose that these cards are sold as individually wrapped random cards and all cards are equally common. You can't see what a card is until after you buy it.

   (a) Approximately how many cards would you expect to need to buy before you have all 150?

   (b) Now suppose you have a nice older brother who still has 25 of his original cards (all unique). If he gives those cards to you, what is the expected number of cards you need to buy now?

   (c) Your older brother isn't that nice; he won't give you his cards for free but he will sell them. If new cards go for \$2.50 a piece, how much would you be willing to pay your brother for his cards? Is the per-card price greater or less than \$2.50? Why?

2. **Quadruply-repeated ones**
   We say that a string of bits has k *quadruply-repeated ones* if there are k positions where four consecutive 1's appear in a row. For example, the string 0100111110 has two quadruply-repeated ones.

   What is the expected number of quadruply-repeated ones in a random n-bit string, when $n \geq 3$ and all n-bit strings are equally likely?

3. **Ball in Bins** You are throwing k balls into n bins. Let X_i be the number of balls thrown into bin i.

   (a) What is $E[X_i]$?

   (b) Compute $E[X_i^2]$. Show that it is at most 2 when $k = n$.

   (c) What is the expected number of empty locations?

   (d) What is the expected number of collisions?

4. **How Many Aces?**
   After you solve this question, you should understand why indicators are so useful! Suppose that you draw 3 cards from a standard deck. Let X denote the number of aces you draw.

   (a) What is $\Pr(X = 0)$?

   (b) What is $\Pr(X = 1)$?

   (c) What is $\Pr(X = 2)$?

   (d) What is $\Pr(X = 3)$?

   (e) Check your answers to make sure they add up to 1.

   (f) Using your previous answers, compute $E(X)$ from the definition.

(g) Define indicators X_i, $1 \le i \le 3$, where X_i is the indicator that the ith card is an ace. Compute $E(X)$ using linearity of expectation.

(h) Are the X_i indicators independent? How does this affect your answer to part (g)?

5. **More Aces in a Deck**

There are four aces in a deck. Suppose you shuffle the deck; define the random variables

$$X_1 = \text{number of non-ace cards before the first ace}$$
$$X_2 = \text{number of non-ace cards between the first and second ace}$$
$$X_3 = \text{number of non-ace cards between the second and third ace}$$
$$X_4 = \text{number of non-ace cards between the third and fourth ace}$$
$$X_5 = \text{number of non-ace cards after the fourth ace}$$

(a) What is $X_1 + X_2 + X_3 + X_4 + X_5$?

(b) Argue that the X_i random variables all have the same distribution.

(c) Use the results of the previous parts to compute $E(X_1)$.

6. **Poisson Distribution**

(a) It is fairly reasonable to model the number of customers entering a shop during a particular hour as a Poisson random variable. Assume that this Poisson random variable X has mean λ. Suppose that whenever a customer enters the shop they leave the shop without buying anything with probability p. Assume that customers act independently, i.e. you can assume that they each simply flip a biased coin to decide whether to buy anything at all. Let us denote the number of customers that buy something as Y and the number of them that do not buy anything as Z (so $X = Y + Z$). What is the probability that $Y = k$ for a given k? How about $\Pr[Z = k]$? Prove that Y and Z are Poisson random variables themselves.
Hint: you can use the identity $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

(b) Assume that you were given two independent Poisson random variables X_1, X_2. Assume that the first has mean λ_1 and the second has mean λ_2. Prove that $X_1 + X_2$ is a Poisson random variable with mean $\lambda_1 + \lambda_2$. Hint: use the results from the previous parts.

7. **Geometric Distribution**

Two faulty machines, M_1 and M_2, are repeatedly run synchronously in parallel (i.e., both machines execute one run, then both execute a second run, and so on). On each run, M_1 fails with probability p_1 and M_2 fails with probability p_2, all failure events being independent. Let the random variables X_1, X_2 denote the number of runs until the first failure of M_1, M_2 respectively; thus X_1, X_2 have geometric distributions with parameters p_1, p_2 respectively. Let X denote the number of runs until the first failure of *either* machine. Show that X also has a geometric distribution, with parameter $p_1 + p_2 - p_1 p_2$.