CS 70 Discrete Mathematics and Probability Theory
Fall 2016 Seshia and Walrand **HW 12**

1 Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of hw party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

2 Problems

1. **Projection Property**

   Use the Projection Property to answer the following questions.

   (a) Prove or disprove: for any function ϕ, $E[(E[Y \mid X])\phi(X)] = 0$.

   (b) Prove or disprove: $E[(Y - E[Y \mid X])L[Y \mid X]] = 0$.

   (c) Prove that the constant c which minimizes $E[(X-c)^2]$ is $c = E[X]$. Use the fact that $E[X - E[X]] = 0$. (*Note*: Although it is possible to directly minimize $E[(X-c)^2]$ by differentiating, we would like you to try to emulate the proofs that the LLSE/MMSE minimize the mean-squared error.)

   (d) Prove the following: $E[X^2 \mid Y] = E[(X - E[X \mid Y])^2 \mid Y] + (E[X \mid Y])^2$. (*Hint*: In the expression $E[X^2 \mid Y]$, try replacing X with $X - E[X \mid Y] + E[X \mid Y]$.)

   (e) Use the result above to compute $E[X^2]$. (Use the law of iterated expectation.)

   (f) We have already shown that $E[E[Y \mid X]] = E[Y]$. Prove that $E[L[Y \mid X]] = E[Y]$.

2. **Quadratic Regression**

   In this question, we will find the best quadratic estimator of Y given X. First, some notation: let μ_i be the ith moment of X, i.e. $\mu_i = E[X^i]$. Also, define $\beta_1 = E[XY]$ and $\beta_2 = E[X^2Y]$. For simplicity, we will assume that $E[X] = E[Y] = 0$ and $E[X^2] = E[Y^2] = 1$. (Note that this poses no loss of generality, because we can always transform the random variables by subtracting their means and dividing by their standard deviations.) We claim that the best quadratic estimator of Y given X is

   $$\hat{Y} = \frac{1}{\mu_3^2 - \mu_4 + 1}(aX^2 + bX + c)$$

   where

   $$\begin{aligned} a &= \mu_3\beta_1 - \beta_2 \\ b &= (1-\mu_4)\beta_1 + \mu_3\beta_2 \\ c &= -\mu_3\beta_1 + \beta_2 \end{aligned}$$

   Your task is to prove the Projection Property for $\hat{Y}$.

   (a) Prove that $E[Y - \hat{Y}] = 0$.

   (b) Prove that $E[(Y - \hat{Y})X] = 0$.

   (c) Prove that $E[(Y - \hat{Y})X^2] = 0$.

3. **Balls in Bins Estimation**

   We throw $n > 0$ balls into $m \geq 2$ bins. Let X and Y represent the number of balls that land in bin 1 and 2 respectively.

(a) Calculate $E[Y \mid X]$. (*Hint*: Your intuition may be more useful than formal calculations.)

(b) What are $L[Y \mid X]$ and $Q[Y \mid X]$ (where $Q[Y \mid X]$ is the best quadratic estimator of Y given X)? (*Hint*: Your justification should be no more than two or three sentences, no calculations necessary! Think carefully about the meaning of the MMSE.)

(c) Unfortunately, your friend is not convinced by your answer to the previous part. Compute $E[X]$ and $E[Y]$.

(d) Compute $\text{var}(X)$.

(e) Compute $\text{cov}(X,Y)$.

(f) Compute $L[Y \mid X]$ using the formula. Ensure that your answer is the same as your answer to part (b).

4. **Iterated Expectation**

In this question, we will try to achieve more familiarity with the law of iterated expectation.

(a) Which of the following are valid conditional expectations? (No justification necessary.)

i. $E[X \mid Y] = 1$

ii. $E[X \mid Y] = X$

iii. $E[X \mid Y] = Y$

iv. $E[X \mid Y] = Y/\cos(Y)$

v. $E[X \mid Y] = XY$

(b) You lost your phone charger! It will take D days for the new phone charger you ordered to arrive at your house (here, D is a random variable). Suppose that on day i, the amount of battery you lose is B_i, where $E[B_i] = \beta$. Let $B = \sum_{i=1}^{D} B_i$ be the total amount of battery drained between now and when your new phone charger arrives. Apply the law of iterated expectation to show that $E[B] = \beta E[D]$. (Here, the law of iterated expectation has a very clear interpretation: the amount of battery you expect to drain is the average number of days it takes for your phone charger to arrive, multiplied by the average amount of battery drained per day.)

(c) Consider now the setting of independent Bernoulli trials, each with probability of success p. Let S_i be the number of successes in the first i trials. Compute $E[S_m \mid S_n]$. (You will need to consider three cases based on whether $m > n$, $m = n$, or $m < n$. Try using your intuition rather than proceeding by calculations.)

5. **Gambling Woes**

Forest proposes a gambling game to you (uh oh!). Every day, you flip two independent fair coins. If both of the coins come up heads, then your fortune triples on that day. If one coin comes up heads and the other coin comes up tails, then your fortune is cut in half. If both of the coins comes up tails, then game over: you lose all of your money! Forest claims that you can get rich quickly with this scheme, but you decide to calculate some probabilities first.

(a) Let M_0 denote your money at the start of the game, and let M_n denote the amount of money you have at the end of the nth day. Compute $E[M_{n+1} \mid M_n]$.

(b) Use the law of iterated expectation to calculate $E[M_{n+1}]$ in terms of $E[M_n]$. Solve your recurrence to obtain an expression for $E[M_{n+1}]$. Do you think this is a fair game?

(c) Calculate $P(M_n > 0)$. What is the behavior as $n \to \infty$? Would you still play this game?