70: Discrete Math and Probability Theory

Programming + Microprocessors $\equiv$ Superpower!

What are your super powerful programs/processors doing?
  Logic and Proofs!
  Induction $\equiv$ Recursion.

What can computers do?
  Work with discrete objects.
  Discrete Math $\implies$ immense application.

Computers learn and interact with the world?
  E.g. machine learning, data analysis, robotics, ...
  Probability!

See note 1, for more discussion.

Instructors

Instructor: Sanjit Seshia.

Professor of EECS (office: 566 Cory)

Starting 12th year at Berkeley.

PhD: in Computer Science, from Carnegie Mellon University.

Research: Formal Methods (a.k.a. Computational Proof Methods)
applied to cyber-physical systems (e.g. “self-driving” cars), computer security, ...

Taught: 149, 172, 144/244, 219C, EECS149.1x on edX, ...

Instructors

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I was born in Belgium[(1)] and came to Berkeley for my PhD. I have been teaching at UCB since 1982.

My wife and I live in Berkeley. We have two daughters (UC alumni – Go Bears!). We like to ski and play tennis (both poorly). We enjoy classical music and jazz.

My research interests include stochastic systems, networks and game theory.

(1)

Admin

Course Webpage: **http://www.eecs70.org/**

Explains policies, has office hours, homework, midterm dates, etc.

Two midterms, final.
midterm 1 before drop date.
midterm 2 before grade option change.

Questions/Announcements $\implies$ piazza:
piazza.com/berkeley/fall2016/cs70

CS70: Lecture 1. Outline.

Today: Note 1. (Note 0 is background. Do read/skim it.)

The language of proofs!

1. Propositions.
2. Propositional Forms.
3. Implication.
4. Truth Tables
5. Quantifiers
6. More De Morgan's Laws

Propositions: Statements that are true or false.

$\sqrt{2}$ is irrational	**Proposition**	True
2+2 = 4	**Proposition**	True
2+2 = 3	**Proposition**	False
826th digit of pi is 4	**Proposition**	False
Jon Stewart is a good comedian	**Not a Proposition**	
All evens > 2 are unique sums of 2 primes	**Proposition**	False
$4+5$	**Not a Proposition.**	
$x+x$	**Not a Proposition.**	

Again: “value” of a proposition is ... True or False

Propositional Forms.

Put propositions together to make another...

Conjunction ("and"): $P \wedge Q$

"$P \wedge Q$" is True when both P and Q are True . Else False .

Disjunction ("or"): $P \vee Q$

"$P \vee Q$" is True when at least one P or Q is True . Else False .

Negation ("not"): $\neg P$

"$\neg P$" is True when P is False . Else False .

Examples:

$\neg$ "$(2+2=4)$" – a proposition that is ... False

"$2+2=3$" $\bigwedge$ "$2+2=4$" – a proposition that is ... False

"$2+2=3$" $\bigvee$ "$2+2=4$" – a proposition that is ... True

Propositional Forms: quick check!

$P =$ "$\sqrt{2}$ is rational"
$Q =$ "826th digit of pi is 2"

P is ...False .
Q is ...True .

$P \wedge Q$ ... False

$P \vee Q$ ... True

$\neg P$ ... True

Put them together..

Propositions:

P_1 - Person 1 rides the bus.

P_2 - Person 2 rides the bus.

....

Suppose we can't have either of the following happen; That either person 1 or person 2 ride the bus and person 3 or 4 ride the bus. Or that person 2 or person 3 ride the bus and that either person 4 ride the bus or person 5 doesn't.

Propositional Form:

$\neg(((P_1 \vee P_2) \wedge (P_3 \vee P_4)) \vee ((P_2 \vee P_3) \wedge (P_4 \vee \neg P_5)))$

Who can ride the bus?

What combinations of people can ride the bus?

This seems ...complicated.

We need a way to keep track!

Truth Tables for Propositional Forms.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

One use for truth tables: Logical Equivalence of propositional forms!

Example: $\neg(P \wedge Q)$ logically equivalent to $\neg P \vee \neg Q$

...because the two propositional forms have the same...

....Truth Table!

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

DeMorgan's Law's for Negation: distribute and flip!

$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ $\qquad$ $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

Implication.

$P \implies Q$ interpreted as

If P, then Q.

True Statements: P, $P \implies Q$.
Conclude: Q is true.

Example: Statement: If you stand in the rain, then you'll get wet.

P = "you stand in the rain"

Q = "you will get wet"

Statement: "Stand in the rain"
Can conclude: "you'll get wet."

Non-Consequences/consequences of Implication

The statement "$P \implies Q$"

only is False if P is True and Q is False .

False implies nothing
P False means Q can be True or False
Anything implies true.
P can be True or False when Q is True

If chemical plant pollutes river, fish die.
If fish die, did chemical plant polluted river?

Not necessarily.

$P \implies Q$ and Q are True does not mean P is True

Instead we have:
$P \implies Q$ and P are True *does* mean Q is True .

Be careful out there!

Some Fun: use propositional formulas to describe implication?
$((P \implies Q) \wedge P) \implies Q$.

Implication and English.

$P \implies Q$

- If P, then Q.
- Q if P.
- P only if Q.
- P is sufficient for Q.
- Q is necessary for P.

Truth Table: implication.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\neg P \vee Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg P \vee Q \quad \equiv P \implies Q.$

These two propositional forms are logically equivalent!

Contrapositive, Converse

- Contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$.
  - If the plant pollutes, fish die.
  - If the fish don't die, the plant does not pollute. (contrapositive)

  - If you stand in the rain, you get wet.
  - If you did not stand in the rain, you did not get wet. (not contrapositive!) converse!
  - If you did not get wet, you did not stand in the rain. (contrapositive.)

  Logically equivalent! Notation: $\equiv$.
  $P \implies Q \equiv \neg P \vee Q \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \implies \neg P$.
- Converse of $P \implies Q$ is $Q \implies P$.
  If fish die the plant pollutes.
  Not logically equivalent!
- **Definition:** If $P \implies Q$ and $Q \implies P$ is P if and only if Q or $P \iff Q$.
  (Logically Equivalent: $\iff$.)

Variables.

Propositions?

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
- $x > 2$
- n is even and the sum of two primes

No. They have a free variable.

We call them predicates, e.g., $Q(x)$ = "x is even"

Same as boolean valued functions from 61A or 61AS!

- $P(n)$ = "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$."
- $R(x)$ = "$x > 2$"
- $G(n)$ = "n is even and the sum of two primes"

Next: Statements about boolean valued functions!!

Quantifiers..

There exists quantifier:

$(\exists x \in S)(P(x))$ means "$P(x)$ is true for some x in S"

Wait! What is S?

S is the **universe**: "the type of x".

Universe examples include..

- $N = \{0, 1, \ldots\}$ (natural numbers).
- $Z = \{\ldots, -1, 0, 1, \ldots\}$ (integers)
- Z^+ (positive integers)
- See note 0 for more!

Quantifiers..

There exists quantifier:

$(\exists x \in S)(P(x))$ means "$P(x)$ is true for some x in S"

For example:

$$(\exists x \in N)(x = x^2)$$

Equivalent to "$(0 = 0) \vee (1 = 1) \vee (2 = 4) \vee \ldots$"

Much shorter to use a quantifier!

For all quantifier;

$(\forall x \in S)\ (P(x))$. means "For all x in S P(x) is True ."

Examples:

"Adding 1 makes a bigger number."

$$(\forall x \in N)\ (x + 1 > x)$$

"the square of a number is always non-negative"

$$(\forall x \in N)(x^2 \geq 0)$$

Quantifiers are not commutative.

- Consider this English statement: ”there is a natural number that is the square of every natural number”, i.e the square of every natural number is the same number!

$$(\exists y \in N)\ (\forall x \in N)\ (y = x^2) \quad \text{False}$$

- Consider this one: ”the square of every natural number is a natural number”...

$$(\forall x \in N)(\exists y \in N)\ (y = x^2) \quad \text{True}$$

Quantifiers....negation...DeMorgan again.

Consider

$$\neg(\forall x \in S)(P(x)),$$

By DeMorgan's law,

$$\neg(\forall x \in S)(P(x)) \iff \exists(x \in S)(\neg P(x)).$$

English: there is an x in S where $P(x)$ does not hold.
What we do in this course! We consider claims.

Claim: $(\forall x)\ P(x)$ "For all inputs x the program works."
For False , find x, where $\neg P(x)$.
Counterexample.
Bad input.
Case that illustrates bug.
For True : prove claim. Next lectures...

Negation of exists.

Consider

$$\neg(\exists x \in S)(P(x))$$

Equivalent to:

$$\neg(\exists x \in S)(P(x)) \iff \forall(x \in S)\neg P(x).$$

English: means that for all x in S , $P(x)$ does not hold.

Which Theorem?

Theorem: $\forall n \in N \left(n \geq 3 \implies \neg(\exists a, b, c \in N\ a^n + b^n = c^n)\right)$

Which Theorem?

Fermat's Last Theorem!

Remember Right-Angled Triangles: for $n = 2$, we have 3,4,5 and 5,7, 12 and ... (Pythagorean triples)

1637: Proof doesn't fit in the margins.

1993: Wiles ...(based in part on Ribet's Theorem)

DeMorgan Restatement:
Theorem: $\neg\left(\exists n \in N\ \exists a, b, c \in N\ (n \geq 3 \wedge a^n + b^n = c^n)\right)$

Summary.

Propositions are statements that are true or false.

Propositional forms use $\wedge, \vee, \neg$.

The meaning of a propositional form is given by its truth table.

Logical equivalence of forms means same truth tables.

Implication: $P \implies Q \iff \neg P \vee Q$.

Contrapositive: $\neg Q \implies \neg P$
Converse: $Q \implies P$

Predicates: Statements with "free" variables.

Quantifiers: $\forall x\ P(x)$, $\exists y\ Q(y)$

Now can state theorems! And disprove false ones!

DeMorgans Laws: "Flip and Distribute negation"
$\neg(P \vee Q) \iff (\neg P \wedge \neg Q)$
$\neg \forall x\ P(x) \iff \exists x\ \neg P(x)$.

Next Time: proofs!