Today

Review for Midterm.

Propositional logic.

A proposition is a statement that is true or false.

Propositions?
$3 = 4 - 1$? Proposition!
$3 = 5$? Proposition!
3 ? Not a proposition!
$n = 3$? Not a proposition...but a predicate.

Predicate: Statement with free variable(s).
Example: $x = 3$ Given a value for x, becomes a proposition.
Predicate?
$n > 3$? Predicate: $P(n)$!
$x = y$? Predicate: $P(x,y)$!
$x + y$? No. An expression, not a proposition.

Quantifiers:
$(\forall x)\ P(x)$. For every x, $P(x)$ is true.
$(\exists x)\ P(x)$. There exists an x, where $P(x)$ is true.
When all variables are quantified, the statement turns into a proposition.

$(\forall n \in N), n^2 \geq n$. $(\forall x \in R)(\exists y \in R) y > x$.

Connecting Propositions with Boolean Operators

$A \wedge B$, $A \vee B$, $\neg A$, $A \implies B$.

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Proofs: truth table or manipulation of known formulas.

Boolean simplification rules - De Morgan's law, commutativity, associativity, etc.

$(\forall x)(P(x) \wedge Q(x)) \equiv (\forall x)P(x) \wedge (\forall x)Q(x)$

Proofs!

Direct: $P \implies Q$
Example: a is even $\implies a^2$ is even.
Approach: What is even? $a = 2k$
$a^2 = 4k^2$.
What is even?
$a^2 = 2(2k^2)$
Integers closed under multiplication!
a^2 is even.

Contrapositive: $P \implies Q$ or $\neg Q \implies \neg P$.
Example: a^2 is odd $\implies a$ is odd.
Contrapositive: a is even $\implies a^2$ is even.

Contradiction: P
$\neg P \implies$ **false**
$\neg P \implies R \wedge \neg R$
Useful for prove something does not exist:
Example: rational representation of $\sqrt{2}$ does not exist.
Example: finite set of primes does not exist. Example: rogue couple does not exist.

Induction.

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n)$.

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step:

$$\begin{aligned} 3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1) - 1 \\ &= 72d + 8 \\ &= 8(9d+1) \end{aligned}$$

Divisible by 8.

□

Stable Marriage: a study in definitions and WOP.

n-men, n-women.

Each person has completely ordered preference list
contains every person of opposite gender.

Pairing.
Set of pairs (m_i, w_j) containing all people *exactly* once.
How many pairs? n.
People in pair are **partners** in pairing.

Rogue Couple in a pairing.
A m_j and w_k who like each other more than their current partners

Stable Pairing.
Pairing with no rogue couples.

Does stable pairing exist?

No, for roommates problem.

Stable Marriage Algorithm (SMA).

(Also called Traditional Marriage Algorithm)

Each Day:
Every man proposes to favorite woman who has not yet rejected him.
Every woman rejects all but best of the men who propose.

Useful Definitions:
Man **crosses off** woman who rejected him.
Woman's current proposer is **"on string."**

"Propose and Reject." : Either men propose or women. But not both.
Traditional propose and reject where men propose.

Key Property: Improvement Lemma:
Every day, if man on string for woman, any future man on string is better.

Stability:
No rogue couple.
suppose rogue couple (M,W) $\implies$ M proposed to W
$\implies$ W ended up with someone she liked better than M.
Not rogue couple!

Optimality/Pessimal

Optimal partner if best partner in any stable pairing.
Not necessarily first in list.
Possibly no stable pairing with that partner.

Man-optimal pairing is pairing where every man gets optimal partner.

Thm: SMA produces male optimal pairing, S.

Man optimal $\implies$ Woman pessimal.
Woman optimal $\implies$ Man pessimal.

Graph Theory!

$G = (V, E)$
V - set of vertices.
$E \subseteq V \times V$ - set of edges.
Focus on simple graphs (at most one edge from a vertex to another)

Undirected: no ordering to edge. Directed: ordered pair of vertices.

Adjacent, Incident, Degree.
In-degree, Out-degree.

Thm: Sum of degrees is $2|E|$.

Pair of Vertices are Connected:
If there is a (simple) path between them.
Related notions: cycle, walk, tour

Connected Component: maximal set of connected vertices.

Connected Graph: one connected component.

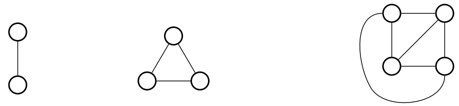
Graph Algorithm: Eulerian Tour

Thm: Every connected graph where every vertex has even degree has an Eulerian Tour; a tour which visits every edge exactly once.

Algorithm:
Take a walk.
Property: return to starting point.
Proof Idea: Even degree.

Recurse on connected components.
Put together.
Property: walk visits every component.
Proof Idea: Original graph connected.

Graph Types: Complete Graph.

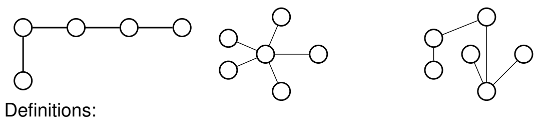


K_n, $|V| = n$

every edge present.
degree of vertex? $|V| - 1$.

Very connected.
Lots of edges: $n(n-1)/2$.

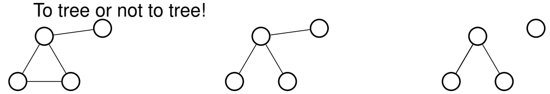
Trees.



Definitions:

A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.
A connected graph where any edge removal disconnects it.
A connected acyclic graph where any edge addition creates a cycle.

To tree or not to tree!

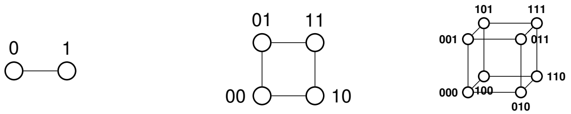


Minimally connected, minimum number of edges to connect.

Hypercube

Hypercubes. Really connected. $|V|\log|V|$ edges!
Also represents bit-strings nicely.

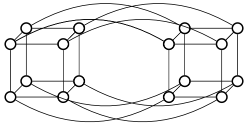
$G=(V,E)$
$|V|=\{0,1\}^n$,
$|E|=\{(x,y)|x$ and y differ in one bit position.$\}$



Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An n-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x,1x)$.



Hypercube:properties

Hamiltonian (Rudrata) Cycle: cycle that visits every node.
Eulerian? If n is even.

Large Cuts: Cutting off k nodes needs $\geq k$ edges.
"Best" cut? Cut apart subcubes: cuts off 2^n nodes with 2^{n-1} edges.

...Modular Arithmetic...

Arithmetic modulo m.
Elements of equivalence classes of integers.
$\{0,\ldots,m-1\}$
and integer $i\equiv a \pmod m$
if $i=a+km$ for integer k.
or if the remainder of i divided by m is a.

Can do calculations by taking remainders
at the beginning,
in the middle
or at the end.

$58+32=90=6 \pmod 7$
$58+32=2+4=6 \pmod 7$
$58+32=2+-3=-1=6 \pmod 7$

Negative numbers work the way you are used to.
$-3=0-3=7-3=4 \pmod 7$

Midterm format

Time: 120 minutes.

Many short answers.
Get at ideas that we study.

Know material well:	fast, correct.
Know material medium:	slower, less correct.
Know material not so well:	Uh oh.

Some longer questions.
Proofs, think about algorithms, properties, etc.

Not so much calculation.

Good Luck!!!