# Today

Review for Midterm.

# Propositional logic.

A proposition is a statement that is true or false.

Propositions?

$3 = 4 - 1$ ? Proposition!

$3 = 5$ ? Proposition!

3 ? Not a proposition!

$n = 3$ ? Not a proposition...but a predicate.

Predicate: Statement with free variable(s).

Example: $x = 3$ Given a value for $x$, becomes a proposition.

Predicate?

$n > 3$ ? Predicate: $P(n)$!

$x = y$? Predicate: $P(x, y)$!

$x + y$? No. An expression, not a proposition.

Quantifiers:

$(\forall x)\ P(x)$. For every $x$, $P(x)$ is true.

$(\exists x)\ P(x)$. There exists an $x$, where $P(x)$ is true.

When all variables are quantified, the statement turns into a proposition.

$(\forall n \in N), n^2 \geq n$. $(\forall x \in R)(\exists y \in R) y > x$.

# Connecting Propositions with Boolean Operators

$A \wedge B$, $A \vee B$, $\neg A$, $A \implies B$.

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Proofs: truth table or manipulation of known formulas.

Boolean simplification rules - De Morgan's law, commutativity, associativity, etc.

$(\forall x)(P(x) \wedge Q(x)) \equiv (\forall x)P(x) \wedge (\forall x)Q(x)$

# Proofs!

Direct: $P \implies Q$

Example: $a$ is even $\implies a^2$ is even.

Approach: What is even? $a = 2k$

$a^2 = 4k^2$.

What is even?

$$a^2 = 2(2k^2)$$

Integers closed under multiplication!

$a^2$ is even.

Contrapositive: $P \implies Q$ or $\neg Q \implies \neg P$.

Example: $a^2$ is odd $\implies a$ is odd.

Contrapositive: $a$ is even $\implies a^2$ is even.

Contradiction: $P$

$\neg P \implies$ **false**

$\neg P \implies R \wedge \neg R$

Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes does not exist. Example: rogue couple does not exist.

# Induction.

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

**Thm:** For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on $n$.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: True for some $n$.
$(3^{2n} - 1 = 8d)$

Induction Step:

$$
\begin{aligned}
3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\
&= 9(8d+1) - 1 \\
&= 72d + 8 \\
&= 8(9d+1)
\end{aligned}
$$

Divisible by 8.

$\square$

# Stable Marriage: a study in definitions and WOP.

$n$-men, $n$-women.

Each person has completely ordered preference list
contains every person of opposite gender.

**Pairing.**
Set of pairs $(m_i, w_j)$ containing all people *exactly* once.
How many pairs? $n$.
People in pair are **partners** in pairing.

**Rogue Couple in a pairing.**
A $m_j$ and $w_k$ who like each other more than their current partners

**Stable Pairing.**
Pairing with no rogue couples.

Does stable pairing exist?

No, for roommates problem.

# Stable Marriage Algorithm (SMA).

(Also called Traditional Marriage Algorithm)

**Each Day:**
**Every man proposes to favorite woman who has not yet rejected him.**
**Every woman rejects all but best of the men who propose.**

Useful Definitions:
Man **crosses off** woman who rejected him.
Woman's current proposer is **"on string."**

"Propose and Reject." : Either men propose or women. But not both.
Traditional propose and reject where men propose.

Key Property: Improvement Lemma:
Every day, if man on string for woman, any future man on string is better.

Stability:
No rogue couple.
suppose rogue couple (M,W) $\implies$ M proposed to W
$\implies$ W ended up with someone she liked better than $M$.
Not rogue couple!

# Optimality/Pessimal

Optimal partner if best partner in any stable pairing.
 Not necessarily first in list.
 Possibly no stable pairing with that partner.

Man-optimal pairing is pairing where every man gets optimal partner.

**Thm:** SMA produces male optimal pairing, $S$.

 Man optimal $\implies$ Woman pessimal.
 Woman optimal $\implies$ Man pessimal.

# Graph Theory!

$G = (V, E)$
$V$ - set of vertices.
$E \subseteq V \times V$ - set of edges.
Focus on simple graphs (at most one edge from a vertex to another)

Undirected: no ordering to edge. Directed: ordered pair of vertices.

Adjacent, Incident, Degree.
In-degree, Out-degree.

**Thm:** Sum of degrees is $2|E|$.

Pair of Vertices are Connected:
If there is a (simple) path between them.
Related notions: cycle, walk, tour

Connected Component: maximal set of connected vertices.

Connected Graph: one connected component.

# Graph Algorithm: Eulerian Tour

**Thm:** Every connected graph where every vertex has even degree has an Eulerian Tour; a tour which visits every edge exactly once.

Algorithm:

Take a walk.

**Property:** return to starting point.

Proof Idea: Even degree.

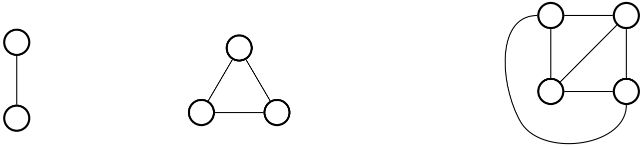
Recurse on connected components.

Put together.

**Property:** walk visits every component.

Proof Idea: Original graph connected.

# Graph Types: Complete Graph.

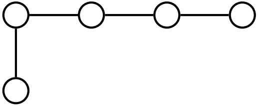
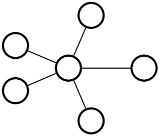
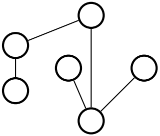


$K_n$, $|V| = n$

every edge present.
degree of vertex? $|V|-1$.

Very connected.
Lots of edges: $n(n-1)/2$.

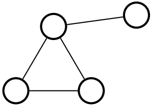
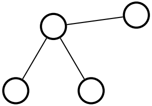
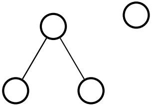
# Trees.

Definitions:

A connected graph without a cycle.
A connected graph with $|V|-1$ edges.
A connected graph where any edge removal disconnects it.
A connected acyclic graph where any edge addition creates a cycle.

To tree or not to tree!

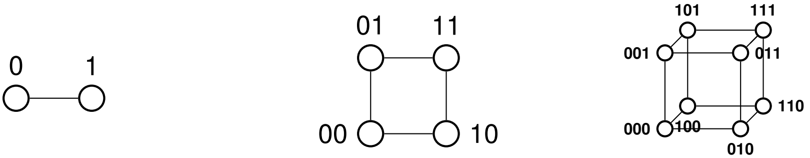




Minimally connected, minimum number of edges to connect.

# Hypercube

Hypercubes. Really connected. $|V|\log|V|$ edges!
Also represents bit-strings nicely.

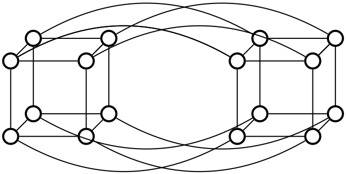
$G = (V, E)$
$|V| = \{0, 1\}^n$,
$|E| = \{(x, y) | x$ and $y$ differ in one bit position.$\}$



# Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$.



# Hypercube:properties

Hamiltonian (Rudrata) Cycle: cycle that visits every node.
Eulerian? If $n$ is even.

Large Cuts: Cutting off $k$ nodes needs $\geq k$ edges.
"Best" cut? Cut apart subcubes: cuts off $2^n$ nodes with $2^{n-1}$ edges.

# ...Modular Arithmetic...

Arithmetic modulo $m$.
  Elements of equivalence classes of integers.
    $\{0, \ldots, m-1\}$
    and integer $i \equiv a \pmod{m}$
      if $i = a + km$ for integer $k$.
       or if the remainder of $i$ divided by $m$ is $a$.

Can do calculations by taking remainders
  at the beginning,
    in the middle
        or at the end.

$58 + 32 = 90 = 6 \pmod{7}$
$58 + 32 = 2 + 4 = 6 \pmod{7}$
$58 + 32 = 2 + -3 = -1 = 6 \pmod{7}$

Negative numbers work the way you are used to.
  $-3 = 0 - 3 = 7 - 3 = 4 \pmod{7}$

# Midterm format

Time: 120 minutes.

Many short answers.
Get at ideas that we study.
Know material well: fast, correct.
Know material medium: slower, less correct.
Know material not so well: Uh oh.

Some longer questions.
Proofs, think about algorithms, properties, etc.

Not so much calculation.

## Good Luck!!!