Outline for next 2 lectures.

1. Cryptography $\Rightarrow$ relation to Bijections
2. Public Key Cryptography
3. RSA system
   3.1 Efficiency: Repeated Squaring.
   3.2 Correctness: Fermat's Little Theorem.
   3.3 Construction.

Cryptography ...

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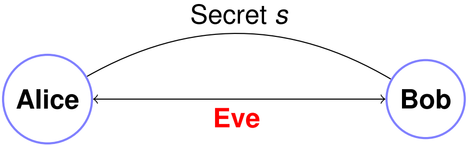

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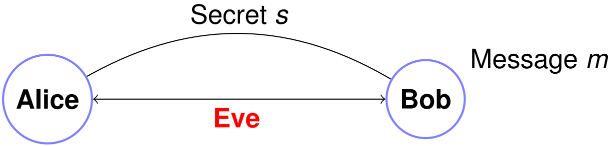

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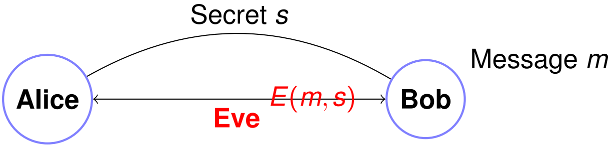

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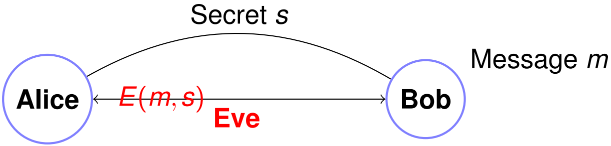

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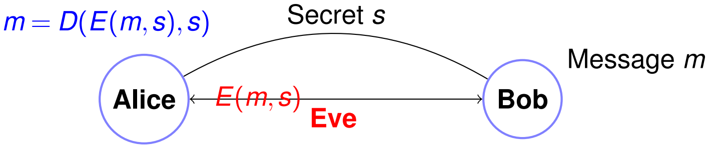

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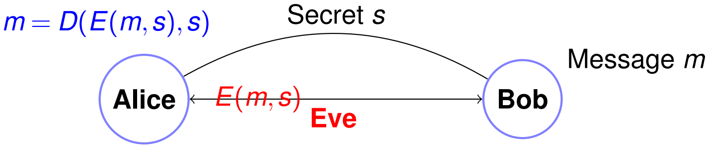

What is the relation between D and E (for the same secret s)?

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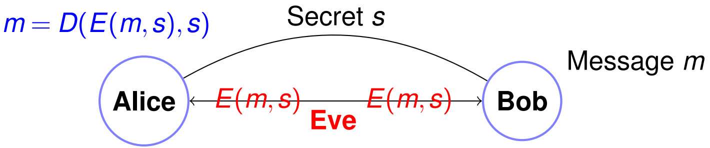

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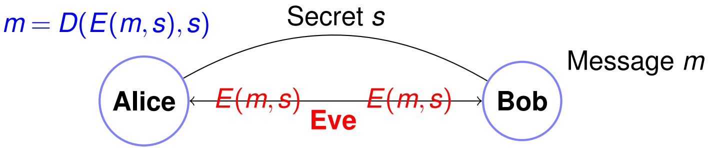

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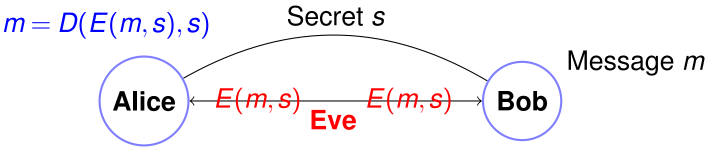

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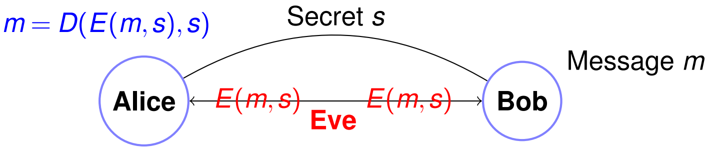

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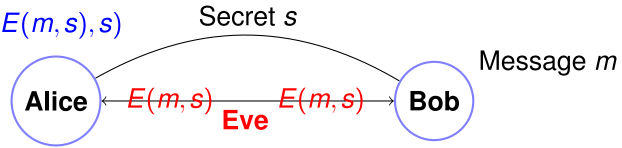

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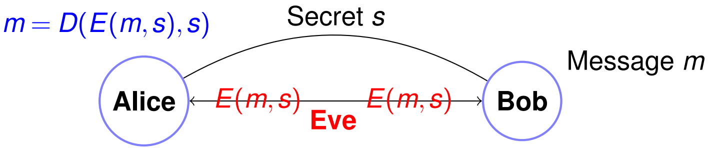

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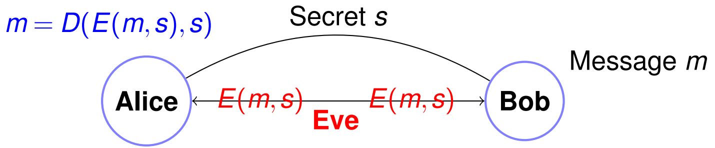

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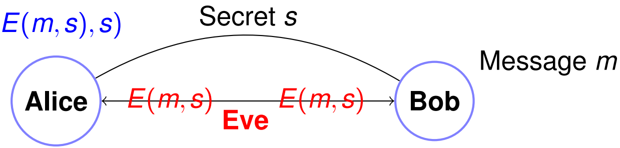

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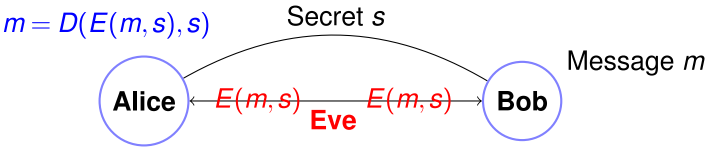

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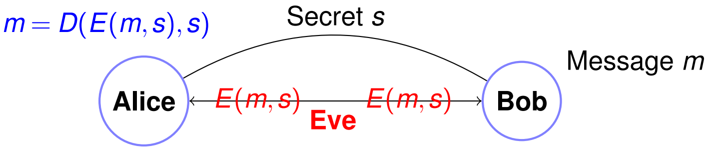

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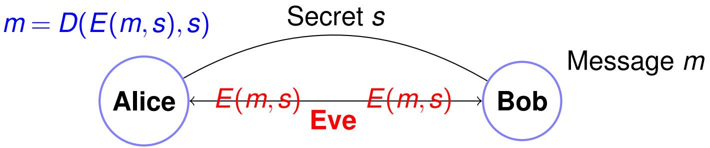

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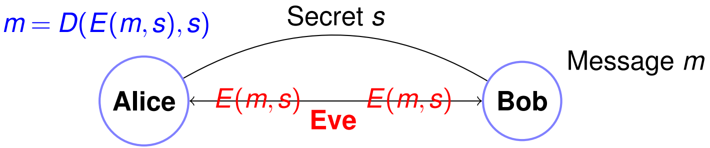

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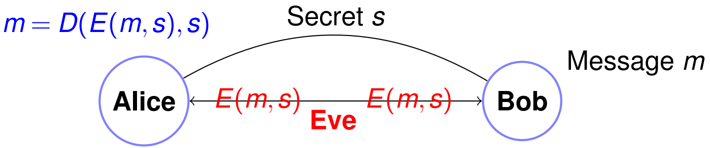

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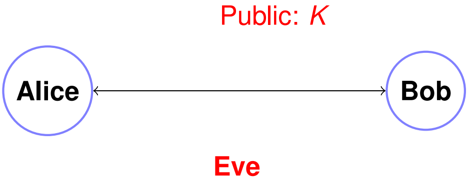

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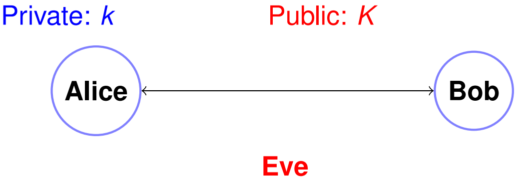

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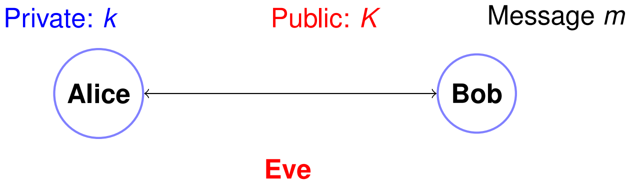

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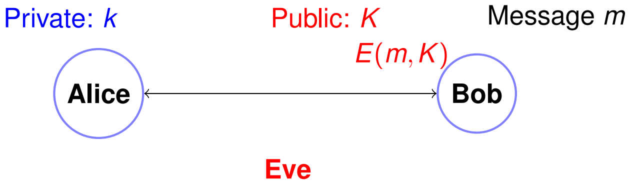

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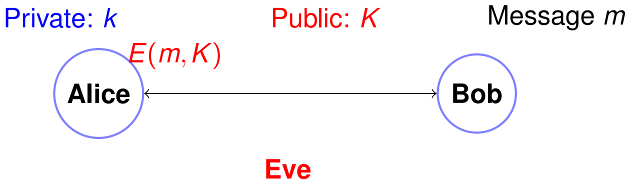

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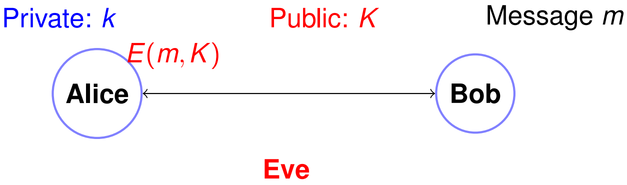

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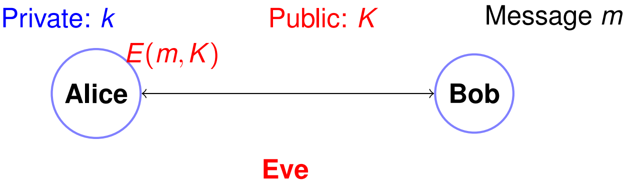

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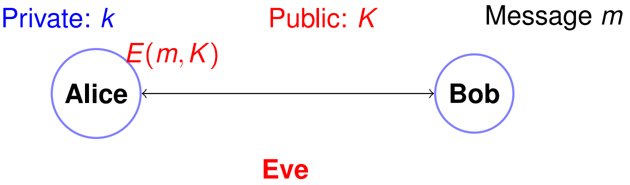

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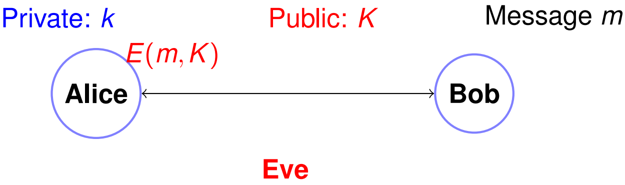

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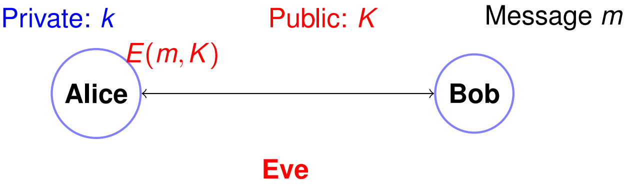

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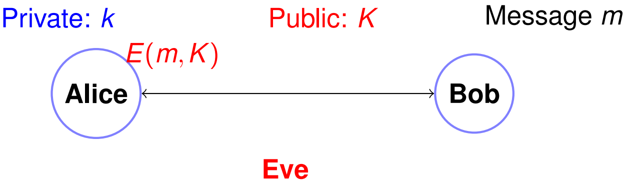

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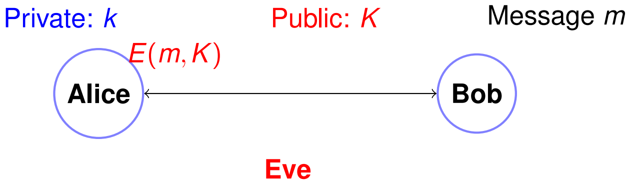

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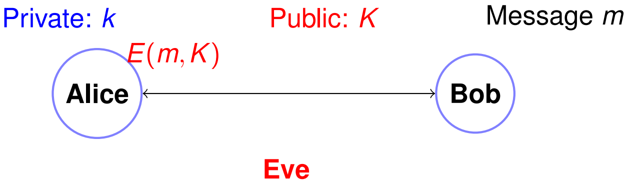

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We don't really know.

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Does $D(E(m)) = m^{ed} = m \mod N$?

[1] Typically small, say $e = 3$.

Is public key crypto unbreakable?

We don't really know.
...but we do it every day!!!

RSA (Rivest, Shamir, and Adleman)
Pick two large primes p and q. Let $N = pq$.
Choose e relatively prime to $(p-1)(q-1)$.[1]
Compute $d = e^{-1} \mod (p-1)(q-1)$. d is the private key!
Announce $N(= p \cdot q)$ and e: $K = (N, e)$ is my public key!

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Example: $p = 7$, $q = 11$.

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Important Considerations

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Q1: Why does RSA work correctly? Fermat's Little Theorem!

Q2: Can RSA be implemented efficiently? Yes, repeated squaring!

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Public Key: $(77, 7)$

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Repeated Squaring to the rescue.

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Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute x^1,

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1. x^y: Compute $x^1, x^2, x^4, \ldots, x^{2^{\lfloor \log y \rfloor}}$.

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Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2, x^4, \ldots, x^{2^{\lfloor \log y \rfloor}}$.
2. Multiply together x^i where the $(\log(i))$th bit of y is 1.

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Fermat's Little Theorem: For prime p, and $a \not\equiv 0 \pmod{p}$,

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$$(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \mod p,$$

Since multiplication is commutative.

$$a^{(p-1)}(1 \cdots (p-1)) \equiv (1 \cdots (p-1)) \mod p.$$

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$\square$