# Today.

Polynomials.

Secret Sharing.

# A secret!

I have a secret!

A number from 0 to 10.

What is it?

Any one of you knows nothing!
Any two of you can figure it out!

Example Applications:
Nuclear launch: need at least 3 out of 5 people to launch!
Cloud service backup: several vendors, each knows nothing.
data from any 2 to recover data.

# Secret Sharing.

**Share secret among $n$ people.**

**Secrecy:** Any $k-1$ knows nothing.
**Roubustness:** Any $k$ knows secret.
**Efficient:** minimize storage.

# Polynomials

A **polynomial**

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

$P(x)$ **contains** point $(a, b)$ if $b = P(a)$.

**Polynomials over reals**: $a_1, \ldots, a_d \in \Re$, use $x \in \Re$.

**Polynomials** $P(x)$ **with arithmetic modulo** $p$: [1] $a_i \in \{0, \ldots, p-1\}$ and

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0 \pmod{p},$$

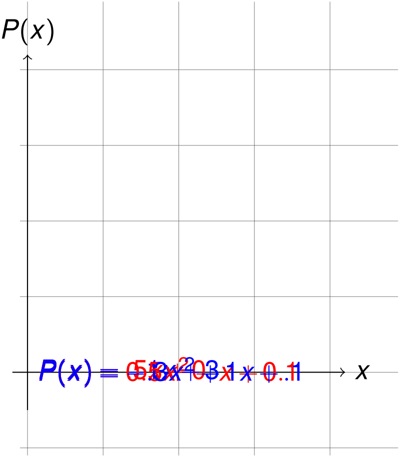
for $x \in \{0, \ldots, p-1\}$.

---

[1] A field is a set of elements with addition and multiplication operations, with inverses. $GF(p) = (\{0, \ldots, p-1\}, + \pmod{p}, * \pmod{p})$.

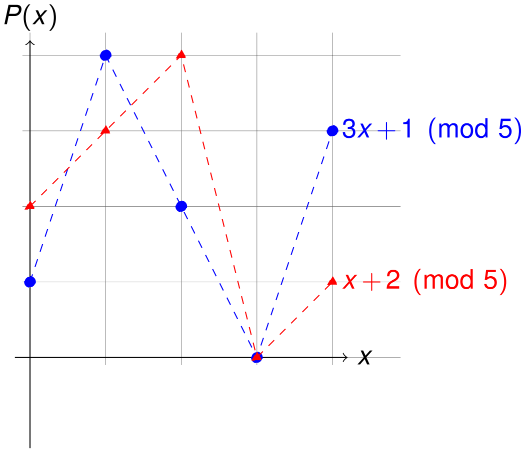
# Polynomial: $P(x) = a_d x^4 + \cdots + a_0$

Line: $P(x) = a_1 x + a_0 = mx + b$




Parabola: $P(x) = a_2 x^2 + a_1 x + a_0 = ax^2 + bx + c$

# Polynomial: $P(x) = a_d x^4 + \cdots + a_0 \pmod{p}$




Finding an intersection.
$x+2 \equiv 3x+1 \pmod 5$
$\implies 2x \equiv 1 \pmod 5 \implies x \equiv 3 \pmod 5$
3 is multiplicative inverse of 2 modulo 5.
Good when modulus is prime!!

# Two points make a line.

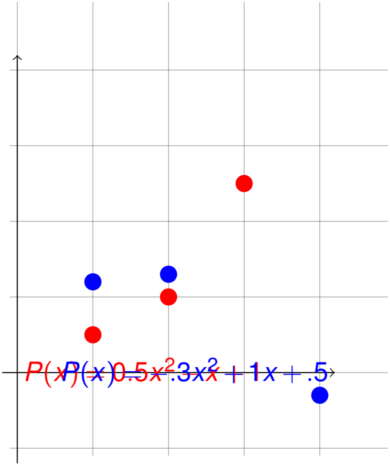
**Fact:** Exactly 1 degree $\leq d$ polynomial contains $d+1$ points. [2]

Two points specify a line. $d=1$, $1+1$ is 2!
Three points specify a parabola. $d=2$, $2+1=3$.

**Modular Arithmetic Fact:** Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime $p$ contains $d+1$ pts.

[2] Points with different $x$ values.

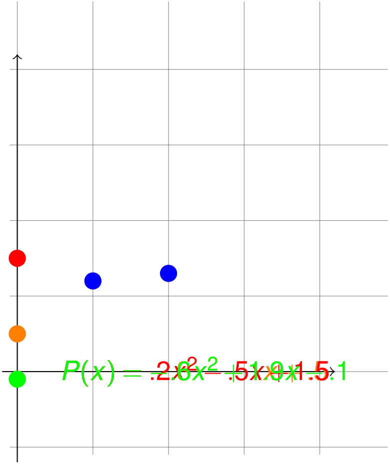
# 3 points determine a parabola.



**Fact:** Exactly 1 degree $\leq d$ polynomial contains $d + 1$ points. [3]

[3] Points with different $x$ values.

# 2 points not enough.



There is $P(x)$ contains blue points and *any* $(0, y)$!

# Modular Arithmetic Fact and Secrets

**Modular Arithmetic Fact:** Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime $p$ contains $d+1$ pts.

**Shamir's $k$ out of $n$ Scheme:**
Secret $s \in \{0, \ldots, p-1\}$

1. Choose $a_0 = s$, and randomly $a_1, \ldots, a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share $i$ is point $(i, P(i) \mod p)$.

**Roubustness:** Any $k$ shares gives secret.
Knowing $k$ pts $\implies$ only one $P(x)$ $\implies$ evaluate $P(0)$.
**Secrecy:** Any $k-1$ shares give nothing.
Knowing $\leq k-1$ pts $\implies$ any $P(0)$ is possible.

# What's my secret?

Remember:
Secret: number from 0 to 10.
  Any one of you knows nothing!
  Any two of you can figure it out!

Shares: points on a line.
Secret: $y$-intercept.
Arithmetic Modulo 11.

What's my secret?

# From $d+1$ points to degree $d$ polynomial?

For a line, $a_1x + a_0 = mx + b$ contains points $(1,3)$ and $(2,4)$.

$$\begin{aligned} P(1) = m(1) + b &\equiv m + b \equiv 3 \pmod 5 \\ P(2) = m(2) + b &\equiv 2m + b \equiv 4 \pmod 5 \end{aligned}$$

Subtract first from second..

$$\begin{aligned} m + b &\equiv 3 \pmod 5 \\ m &\equiv 1 \pmod 5 \end{aligned}$$

Backsolve: $b \equiv 2 \pmod 5$. Secret is 2.

And the line is...

$$x + 2 \mod 5.$$

# What's my secret?

$$P(1) = m(1) + b \equiv 5 \pmod{11}$$
$$P(3) = m(3) + b \equiv 9 \pmod{11}$$

Subtract first from second.

$$2m \equiv 4 \pmod{11}$$

Multiplicative inverse of 2 $\pmod{11}$ is 6: $6 \times 2 \equiv 12 \equiv 1 \pmod{11}$
Multiply both sides by 6.

$$12m = 24 \pmod{11}$$
$$m = 2 \pmod{11}$$

Backsolve: $2 + b \equiv 5 \pmod{11}$. Or $\quad b = 3 \pmod{11}$.

Secret is 3.

# Quadratic

For a quadratic polynomial, $a_2x^2 + a_1x + a_0$ hits $(1,2);(2,4);(3,0)$.
Plug in points to find equations.

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 5 \\ P(3) = 4a_2 + 3a_1 + a_0 &\equiv 0 \pmod 5 \end{aligned}$$

$$\begin{aligned} a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ 3a_1 + 2a_0 &\equiv 1 \pmod 5 \\ 4a_1 + 2a_0 &\equiv 2 \pmod 5 \end{aligned}$$

Subtracting 2nd from 3rd yields: $a_1 = 1$.
$a_0 = (2 - 4(a_1))2^{-1} = (-2)(2^{-1}) = (3)(3) = 9 \equiv 4 \pmod 5$
$a_2 = 2 - 1 - 4 \equiv 2 \pmod 5$ .

So polynomial is $2x^2 + 1x + 4 \pmod 5$

# In general: Linear System.

Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)$.

Solve...

$$
\begin{aligned}
a_{k-1}x_1^{k-1} + \cdots + a_0 &\equiv y_1 \pmod{p} \\
a_{k-1}x_2^{k-1} + \cdots + a_0 &\equiv y_2 \pmod{p} \\
&\cdot \\
&\cdot \\
a_{k-1}x_k^{k-1} + \cdots + a_0 &\equiv y_k \pmod{p}
\end{aligned}
$$

Will this always work?

As long as solution **exists** and it is **unique!** And...

**Modular Arithmetic Fact:** Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime $p$ contains $d+1$ pts.

# Another Construction: Interpolation!

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,3);(2,4);(3,0)$.

Find $\Delta_1(x)$ polynomial contains $(1,1);(2,0);(3,0)$.

Try $(x-2)(x-3) \pmod 5$.

Value is 0 at 2 and 3. Value is 2 at 1. Not 1! Doh!!
So "Divide by 2" or multiply by 3.
$\Delta_1(x) = (x-2)(x-3)(3) \pmod 5$ contains $(1,1);(2,0);(3,0)$.

$\Delta_2(x) = (x-1)(x-3)(4) \pmod 5$ contains (1,0);(2,1);(3,0).

$\Delta_3(x) = (x-1)(x-2)(3) \pmod 5$ contains (1,0);(2,0);(3,1 ).

But wanted to hit $(1,3);(2,4);(3,0)$!

$P(x) = 3\Delta_1(x) + 4\Delta_2(x) + 0\Delta_3(x)$ works.

Same as before?

...after a lot of calculations... $P(x) = 2x^2 + 1x + 4 \mod 5$.

The same as before!

# Interpolation: in general.

Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)$.

$$\Delta_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}.$$

Numerator is 0 at $x_j \neq x_i$.

Denominator makes it 1 at $x_i$.

And..

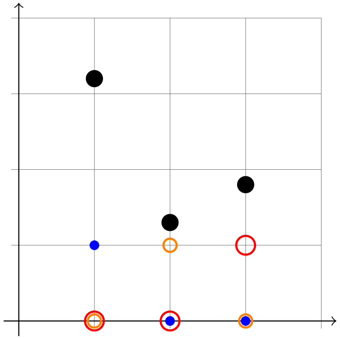
$$P(x) = y_1 \Delta_1(x) + y_2 \Delta_2(x) + \cdots + y_k \Delta_k(x).$$

hits points $(x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)$.

Construction proves the existence of a degree $d$ polynomial!

# Interpolation: in pictures.

Points: $(1, 3.2)$, $(2, 1.3)$, $(3, 1.8)$.



$\Delta_1(x)$ $\Delta_2(x)$ $\Delta_3(x)$

Scale each $\Delta_i$ function and add to contain points.

$$P(x) = 3.2\ \Delta_1(x) + 1.3\Delta_2(x) + 1.8\Delta_3(x)$$

# Interpolation and Existence

Interpolation takes $d+1$ points and produces a degree $d$ polynomial that contains the points.

Construction proves the existence of a degree $d$ polynomial that contains points!

Is it the only degree $d$ polynomial that contains the points?

# Uniqueness.

**Uniqueness Fact.** At most one degree $d$ polynomial hits $d+1$ points.

**Proof:**

**Roots fact:** Any degree $d$ polynomial has at most $d$ roots.

Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.

$R(x) = Q(x) - P(x)$ has $d+1$ roots and is degree $d$.
Contradiction.

$\square$

Must prove **Roots fact.**

## Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x-3)$ modulo 5.

```
              4 x + 4 r 4
         -----------------
x - 3 ) 4x^2 - 3 x + 2
     - (4x^2 - 2 x)
        ----------
              4 x + 2
           - (4 x - 2)
                -------
                    4
```

$4x^2 - 3x + 2 \equiv (x-3)(4x+4) + 4 \pmod 5$

In general, divide $P(x)$ by $(x-a)$ gives $Q(x)$ and remainder $r$.

That is, $P(x) = (x-a)Q(x) + r$

# Only $d$ roots.

**Lemma 1:** $P(x)$ has root $a$ iff $P(x)/(x-a)$ has remainder 0:
$P(x) = (x-a)Q(x)$.

**Proof:** $P(x) = (x-a)Q(x) + r$.
Plugin $a$: $P(a) = r$. It is a root if and only if $r = 0$. □

**Lemma 2:** $P(x)$ has $d$ roots; $r_1, \ldots, r_d$ then
$P(x) = c(x-r_1)(x-r_2)\cdots(x-r_d)$.

**Proof Sketch:** By induction.
Induction Step: $P(x) = (x-r_1)Q(x)$ by Lemma 1.

$P(x) = 0$ if and only if $(x-r_1)$ is 0 or $Q(x) = 0$.
$ab = 0 \implies a = 0$ or $b = 0$ in field.
Root either at $r_1$ or root of $Q(x)$.

$Q(x)$ has smaller degree and $r_2, \ldots r_d$ are roots.
Use the induction hypothesis. □

$d+1$ roots implies degree is at least $d+1$.

**Roots fact:** Any degree $d$ polynomial has at most $d$ roots.

# Finite Fields

Proof works for reals, rationals, and complex numbers.

..but not for integers, since no multiplicative inverses.

Arithmetic modulo a prime $p$ has multiplicative inverses..

..and has only a finite number of elements.

Good for computer science.

Arithmetic modulo a prime $m$ is a **finite field** denoted by $F_m$ or $GF(m)$.

Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.