Lecture 14. Outline.

1. Finish Polynomials and Secrets.
2. Finite Fields: Abstract Algebra
3. Erasure Coding

Modular Arithmetic Fact and Secrets

Modular Arithmetic Fact: There is exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p that contains $d+1$ pts.

Note: The points have to have different x values!

Shamir's k out of n Scheme:
Secret $s \in \{0, \ldots, p-1\}$

1. Choose $a_0 = s$, and random $a_1, \ldots, a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i for $i \geq 1$ is point $(i, P(i) \mod p)$.

Robustness: Any k shares gives secret.
Knowing k pts, find unique $P(x)$, evaluate $P(0)$.
Secrecy: Any $k-1$ shares give nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.

There exists a polynomial...

Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

Proof of at least one polynomial:
Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_{d+1}, y_{d+1})$.

$$\Delta_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}.$$

Numerator is 0 at $x_j \neq x_i$.

Denominator makes it 1 at x_i.

And..

$$P(x) = y_1 \Delta_1(x) + y_2 \Delta_2(x) + \cdots + y_{d+1} \Delta_{d+1}(x).$$

hits points $(x_1, y_1); (x_2, y_2) \cdots (x_{d+1}, y_{d+1})$. Degree d polynomial!

Construction proves the existence of a polynomial!

Reiterating Examples.

$$\Delta_i(x) = \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)}.$$

Degree 1 polynomial, $P(x)$, that contains $(1,3)$ and $(3,4)$?

Work modulo 5.

$\Delta_1(x)$ contains $(1,1)$ and $(3,0)$.

$$\begin{aligned}\Delta_1(x) &= \frac{(x-3)}{1-3} = \frac{x-3}{-2} \\ &= 2(x-3) = 2x-6 = 2x+4 \pmod 5.\end{aligned}$$

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,3);(2,4);(3,0)$.

Work modulo 5.

Find $\Delta_1(x)$ polynomial contains $(1,1);(2,0);(3,0)$.

$$\begin{aligned}\Delta_1(x) &= \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2} = 3(x-2)(x-3) \\ &= 3x^2+1 \pmod 5\end{aligned}$$

Put the delta functions together.

Simultaneous Equations Method.

For a line, $a_1x + a_0 = mx + b$ contains points $(1,3)$ and $(2,4)$.

$$\begin{aligned} P(1) = m(1) + b &\equiv m + b \equiv 3 \pmod 5 \\ P(2) = m(2) + b &\equiv 2m + b \equiv 4 \pmod 5 \end{aligned}$$

Subtract first from second..

$$\begin{aligned} m + b &\equiv 3 \pmod 5 \\ m &\equiv 1 \pmod 5 \end{aligned}$$

Backsolve: $b \equiv 2 \pmod 5$. Secret is 2.

And the line is...

$$x + 2 \mod 5.$$

Quadratic

For a quadratic polynomial, $a_2x^2 + a_1x + a_0$ hits $(1,2);(2,4);(3,0)$.
Plug in points to find equations.

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 5 \\ P(3) = 4a_2 + 3a_1 + a_0 &\equiv 0 \pmod 5 \end{aligned}$$

$$\begin{aligned} a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ 3a_1 + 2a_0 &\equiv 1 \pmod 5 \\ 4a_1 + 2a_0 &\equiv 2 \pmod 5 \end{aligned}$$

Subtracting 2nd from 3rd yields: $a_1 = 1$.
$a_0 = (2 - 4(a_1))2^{-1} = (-2)(2^{-1}) = (3)(3) = 9 \equiv 4 \pmod 5$
$a_2 = 2 - 1 - 4 \equiv 2 \pmod 5$.

So polynomial is $2x^2 + 1x + 4 \pmod 5$

In general..

Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)$.

Solve...

$$\begin{aligned} a_{k-1}x_1^{k-1} + \cdots + a_0 &\equiv y_1 \pmod{p} \\ a_{k-1}x_2^{k-1} + \cdots + a_0 &\equiv y_2 \pmod{p} \\ \vdots \quad & \quad \vdots \quad \vdots \\ a_{k-1}x_k^{k-1} + \cdots + a_0 &\equiv y_k \pmod{p} \end{aligned}$$

Will this always work?

As long as solution **exists** and it is **unique!** And...

Modular Arithmetic Fact: Exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p contains $d+1$ pts.

Summary.

Modular Arithmetic Fact: Exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p contains $d+1$ pts.

Existence:
    Lagrange Interpolation.

Uniqueness: (proved last time)
    At most d roots for degree d polynomial.

Finite Fields

Proof works for reals, rationals, and complex numbers.

..but not for integers, since no multiplicative inverses.

Arithmetic modulo a prime p has multiplicative inverses..

..and has only a finite number of elements.

Good for computer science.

Arithmetic modulo a prime p is a **finite field** denoted by F_p or $GF(p)$.

Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

Secret Sharing Revisited

Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.
Shamir's k out of n Scheme:
Secret $s \in \{0, \ldots, p-1\}$

1. Choose $a_0 = s$, and random $a_1, \ldots, a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i is point $(i, P(i) \mod p)$.

Robustness: Any k knows secret.
Knowing k pts, only one $P(x)$, evaluate $P(0)$.
Secrecy: Any $k-1$ knows nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.
Efficiency: ???

Efficiency.

Need $p > n$ to hand out n shares: $P(1) \ldots P(n)$.

For b-bit secret, must choose a prime $p > 2^b$.

Theorem: There is always a prime between n and $2n$.

Working over numbers within 1 bit of secret size.
Minimal!

With k shares, reconstruct polynomial, $P(x)$.

With $k-1$ shares, any of p values possible for $P(0)$!

(Within 1 bit of) any b-bit string possible!

(Within 1 bit of) b-bits are missing: one $P(i)$.

Within 1 of optimal number of bits.

Runtime.

Runtime: polynomial in k, n, and $\log p$.

1. Evaluate degree $n-1$ polynomial $n+k$ times using $\log p$-bit numbers. $O(kn \log^2 p)$.
2. Reconstruct secret by solving system of n equations using $\log p$-bit arithmetic. $O(n^3 \log^2 p)$.
3. Matrix has special form so $O(n \log n \log^2 p)$ reconstruction.

Faster versions in practice are almost as efficient.

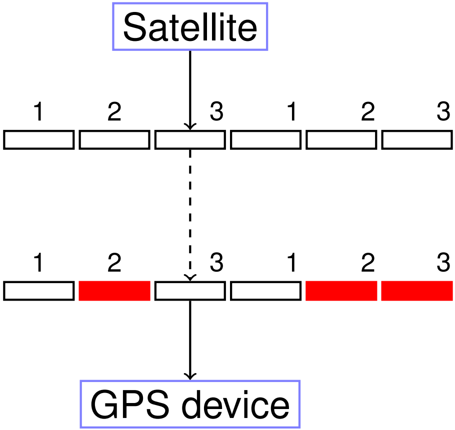
A bit of counting.

What is the number of degree d polynomials over $GF(m)$?

- m^{d+1}: $d+1$ coefficients from $\{0,\ldots,m-1\}$.
- m^{d+1}: $d+1$ points with y-values from $\{0,\ldots,m-1\}$

Infinite number for reals, rationals, complex numbers!

Erasure Codes.



3 packet message. So send 6!

Lose 3 out 6 packets.

Gets packets 1,1,and 3.

Problem: Want to send a message with n packets.

Channel: Lossy channel: loses k packets.

Question: Can you send $n+k$ packets and recover message?

Solution Idea: Use Polynomials!!!

Solution Idea.

n packet message, channel that loses k packets.

Must send $n+k$ packets!

Any n packets should allow reconstruction of n packet message.

Any n point values allow reconstruction of degree $n-1$ polynomial which has n coefficients!

Alright!!!

Use polynomials.

Problem: Want to send a message with n packets.

Channel: Lossy channel: loses k packets.

Question: Can you send $n+k$ packets and recover message?

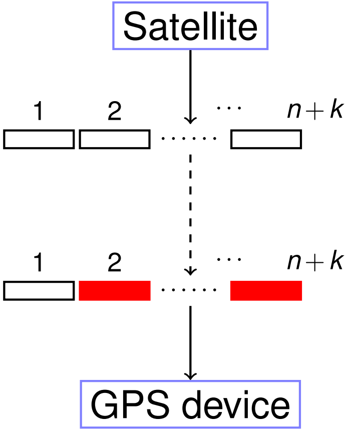
A degree $n-1$ polynomial determined by any n points!

Erasure Coding Scheme: message = $m_0, m_1, m_2, \ldots, m_{n-1}$. Each m_i is a packet.

1. Choose prime $p > 2^b$ for packet size b (size = number of bits).
2. $P(x) = m_{n-1}x^{n-1} + \cdots m_0 \pmod{p}$.
3. Send $P(1), \ldots, P(n+k)$.

Any n of the $n+k$ packets gives polynomial ...and message!

Erasure Codes.



n packet message. So send $n+k$!

Lose k packets.

Any n packets is enough!

n packet message.

Optimal.

Comparison with Secret Sharing.

Comparing information content:

Secret Sharing: each share is size of whole secret.

Coding: Each packet has size $1/n$ of the whole message.

Erasure Code: Example.

Send message of 1,4, and 4. up to 3 erasures. $n = 3, k = 3$

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

How?

Lagrange Interpolation.
Linear System.

Work modulo 5.

$P(x) = x^2 \pmod 5$
$P(1) = 1, P(2) = 4, P(3) = 9 = 4 \pmod 5$

Send $(0, P(0)) \ldots (5, P(5))$.

6 points. Better work modulo 7 at least!

Why? $(0, P(0)) = (5, P(5)) \pmod 5$

Example

Make polynomial with $P(1)=1$, $P(2)=4$, $P(3)=4$.

Modulo 7 to accommodate at least 6 packets.

Linear equations:

$$\begin{aligned} P(1)=a_2+a_1+a_0 &\equiv 1 \pmod 7 \\ P(2)=4a_2+2a_1+a_0 &\equiv 4 \pmod 7 \\ P(3)=2a_2+3a_1+a_0 &\equiv 4 \pmod 7 \end{aligned}$$

$6a_1+3a_0=2 \pmod 7$, $5a_1+4a_0=0 \pmod 7$

$a_1=2a_0$. $a_0=2 \pmod 7$ $a_1=4 \pmod 7$ $a_2=2 \pmod 7$

$P(x)=2x^2+4x+2$

$P(1)=1$, $P(2)=4$, and $P(3)=4$

Send
Packets: $(1,1),(2,4),(3,4),(4,7),(5,2),(6,0)$

Notice that packets contain “x-values”.

Summary: Polynomials are useful!

- ..give Secret Sharing.
- ..give Erasure Codes.

Next time: correct broader class of errors!