Lecture 14. Outline.

1. Finish Polynomials and Secrets.
2. Finite Fields: Abstract Algebra
3. Erasure Coding

Modular Arithmetic Fact and Secrets

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Knowing k pts, find unique $P(x)$, evaluate $P(0)$.
Secrecy: Any $k-1$ shares give nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.

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Construction proves the existence of a polynomial!

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Put the delta functions together.

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Subtract first from second..

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And the line is...

$$x + 2 \mod 5.$$

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Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

Secret Sharing Revisited

Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.
Shamir's k out of n Scheme:
Secret $s \in \{0, \ldots, p-1\}$

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Faster versions in practice are almost as efficient.

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Infinite number for reals, rationals, complex numbers!

Erasure Codes.

Satellite

GPS device

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Satellite

3 packet message.

GPS device

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Lose 3 out 6 packets.

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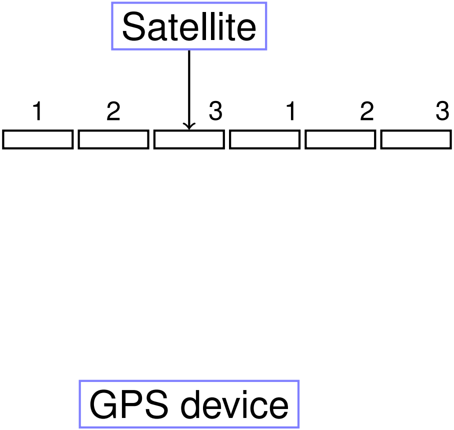

3 packet message. So send 6!

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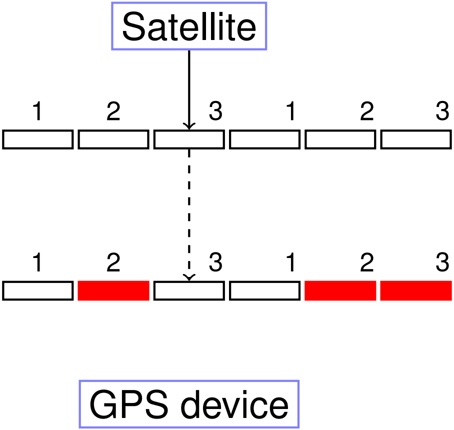

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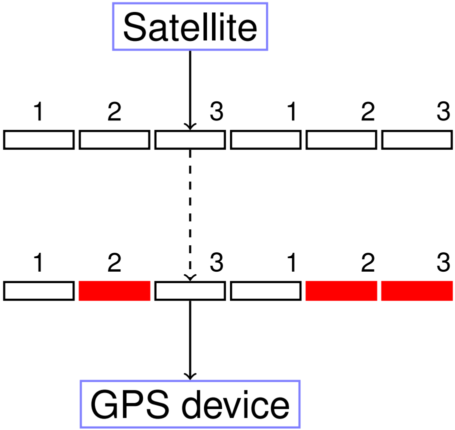

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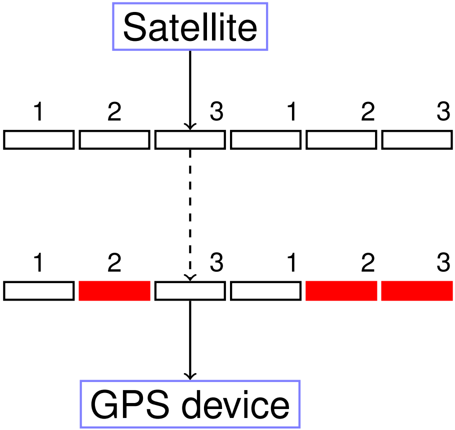

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Gets packets 1,1,and 3.

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Solution Idea: Use Polynomials!!!

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Alright!!!

Use polynomials.

Problem: Want to send a message with n packets.

Channel: Lossy channel: loses k packets.

Question: Can you send $n+k$ packets and recover message?

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Erasure Codes.

Satellite

GPS device

Erasure Codes.

Satellite

n packet message.

GPS device

Erasure Codes.

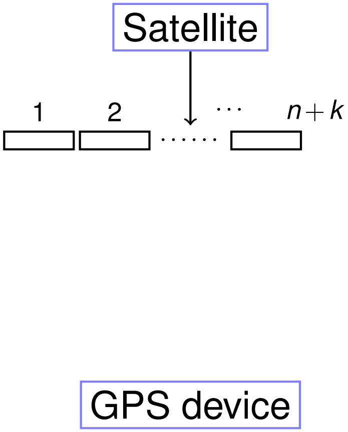
Satellite

n packet message.

Lose *k* packets.

GPS device

Erasure Codes.



n packet message. So send $n+k$!

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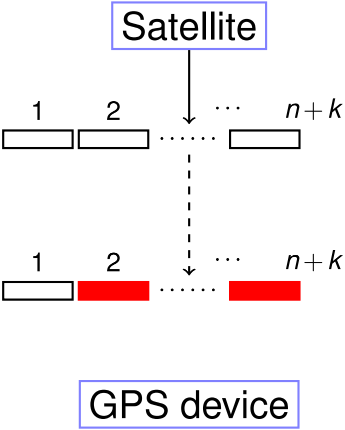

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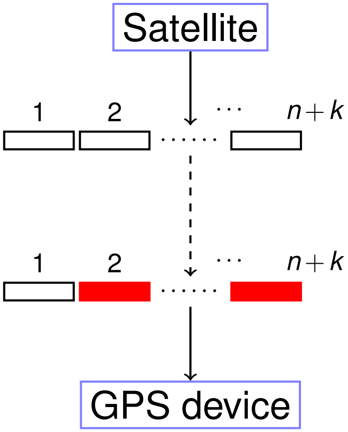

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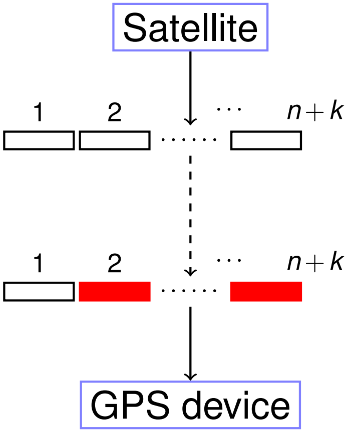

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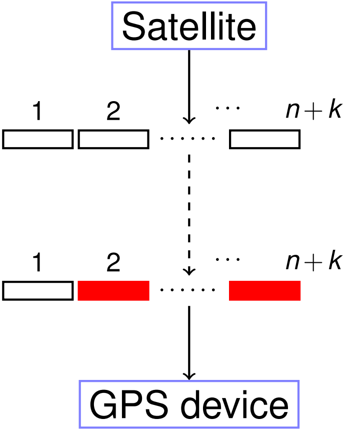

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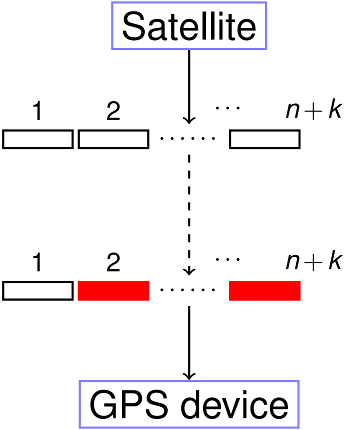

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Optimal.

Comparison with Secret Sharing.

Comparing information content:

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Secret Sharing: each share is size of whole secret.

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Coding: Each packet has size $1/n$ of the whole message.

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Send message of 1,4, and 4.

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Send message of 1,4, and 4. up to 3 erasures.

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$P(x) = 2x^2 + 4x + 2$

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Example

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Modulo 7 to accommodate at least 6 packets.

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$P(x) = 2x^2 + 4x + 2$

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Send

Example

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Modulo 7 to accommodate at least 6 packets.

Linear equations:

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 1 \pmod 7 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 7 \\ P(3) = 2a_2 + 3a_1 + a_0 &\equiv 4 \pmod 7 \end{aligned}$$

$6a_1 + 3a_0 = 2 \pmod 7$, $5a_1 + 4a_0 = 0 \pmod 7$

$a_1 = 2a_0$. $a_0 = 2 \pmod 7$ $a_1 = 4 \pmod 7$ $a_2 = 2 \pmod 7$

$P(x) = 2x^2 + 4x + 2$

$P(1) = 1$, $P(2) = 4$, and $P(3) = 4$

Send
Packets: $(1,1),(2,4),(3,4),(4,7),(5,2),(6,0)$

Example

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

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Notice that packets contain “x-values”.

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Next time: correct broader class of errors!