Infinity and Uncountability.

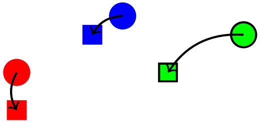
- Countable
- Countably infinite.
- Enumeration

How big is the set of reals or the set of integers?

Infinite!

Is one bigger or smaller?

Same size?



Same number?
Make a function f : Circles $\to$ Squares.
f(red circle) $=$ red square
f(blue circle) $=$ blue square
f(circle with black border) $=$ square with black border
One to one. Each circle mapped to different square.
One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
Onto. Each square mapped to from some circle .
Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

Isomorphism principle: If there is $f : D \to R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
or
$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

Onto: For all $y \in R$, $\exists x \in D, y = f(x)$.

$f(\cdot)$ is a **bijection** if it is one to one and onto.

Isomorphism principle:
If there is a bijection $f : D \to R$ then $|D| = |R|$.

Combinatorial Proofs.

The number of subsets of a set $\{a_1,\ldots,a_n\}$.?

Equal to the number of binary n-bit strings.

f : Subsets. $\rightarrow$ Strings.

$f(x) = (g(x,a_1), g(x,a_2), \ldots, g(x,a_n))$

$$g(x,a) = \begin{cases} 1 & a \in x \\ 0 & \text{otherwise} \end{cases}$$

Example:
$S = \{1,2,3,4,5\}$, $x = \{1,3,4\}$.
$f(x) = (1,0,1,1,0)$.

$|P(S)| = |\{0,1\}^n| = 2^n$.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where's 0?

Which is bigger?
The positive integers, Z^+, or the natural numbers, N.

Natural numbers. $0,1,2,3,\ldots.$

Positive integers. $1,2,3,\ldots.$

Where's 0?

More natural numbers!

Consider $f : Z^+ \rightarrow N$ where $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n,
for $z = n+1$, $f(z) = (n+1) - 1 = n$.
Onto!

Bijection!

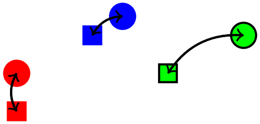
$|Z^+| = |N|$.

But.. but where's zero? "It comes from 1."

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n+1$. $0 \rightarrow 1, 1 \rightarrow 2, \ldots$

Bijection from A to B $\implies$ a bijection from B to A.



Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

$f : N \to E$.

$f(n) \to 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. $e/2$ is natural since e is even
One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y. \equiv f(x) \neq f(y)$

Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$
$\ldots$.

Onto: For any $z \in Z$,
if $z \geq 0$, $f(2z) = z$ and $2z \in N$.
if $z < 0$, $f(2|z| - 1) = z$ and $2|z| + 1 \in N$.

Integers and naturals have same size!

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2
...	...

Notice that: A listing “is” a bijection with a subset of natural numbers.
Function $\equiv$ “Position in list.”
If finite: bijection with $\{0, \ldots, |S| - 1\}$
If infinite: bijection with N.

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
$\ldots$
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, \ldots..\}$
$Z = \{\{0, 1, 2, \ldots,\}$ and then $\{-1, -2, \ldots\}\}$

When do you get to -1? at infinity?

Need to be careful.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Implications:
Z^+ is countable.
It is infinite.
There is a bijection with the natural numbers.
So it is countably infinite.

All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.
ε is empty string.

For any string, it appears at some position in the list.
If n bits, it will appear before position 2^{n+1}.

Should be careful here.

$B = \{\varepsilon; , 0, 00, 000, 0000, ...\}$
Never get to 1.

Fractions?

Can you enumerate the rational numbers in order?

$0, \ldots, 1/2, ..$

Where is $1/2$ in list?

After $1/3$, which is after $1/4$, which is after $1/5$...

A thing about fractions:
any two fractions has another fraction between it.

Does this mean we can't even get to "next" fraction?

Can't list in "order"?

Pairs of natural numbers.

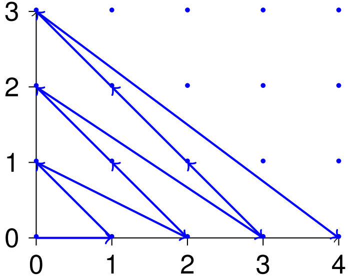
Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

For finite sets S_1 and S_2,
then $S_1 \times S_2$
has size $|S_1| \times |S_2|$.

So, is $N \times N$ countably infinite squared ???

Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



The pair (a,b), is in first $(a+b+1)(a+b)/2$ elements of list!
(i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!

Rationals?

Positive rational number.
Lowest terms: a/b
$a, b \in N$
with $gcd(a,b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ??? No!

Repeatedly and alternatively take one from each list.
  Interleave Streams in 61A

The rationals are countably infinite!

Real numbers..

Is the set of real numbers the “same size” as integers?

The reals.

Are the set of reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1-1/e$
.345212312... Some real number

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct “diagonal” number: $.77677\ldots$

Diagonal Number: Digit *i* is 7 if number *i*’s, *i*th digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0,1]$ is not countable!!

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.

If reals are countable then so must $[0, 1]$.

Diagonalization: Summary

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list
   $\Longrightarrow$ t is not in the list.
5. Show that t is in S.
6. Contradiction.

Another diagonalization.

The set of all subsets of N.

Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.
$\implies$ D is not in the listing.

D is a subset of N.

L does not contain all subsets of N.

Contradiction.

Theorem: The set of all subsets of N is not countable.
(The "set of all subsets of N" is the **powerset** of N.)

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : R^+ \to [0,1]$.

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\implies f(x) \neq f(y)$.
If one is in $[0,1/2]$ and one isn't, different ranges $\implies f(x) \neq f(y)$.
Bijection!

$[0,1]$ is same cardinality as nonnegative reals!

Summary.

- Bijections to equate cardinality of infinite sets
- Countable (infinite) sets
- Uncountable sets
- Diagonalization