Infinity and Uncountability.

- Countable
- Countably infinite.
- Enumeration

How big is the set of reals or the set of integers?

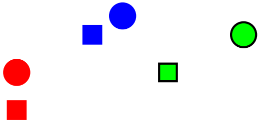
Infinite!

How big is the set of reals or the set of integers?

Infinite!

Is one bigger or smaller?

Same size?



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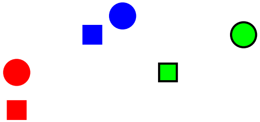

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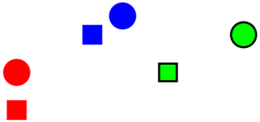
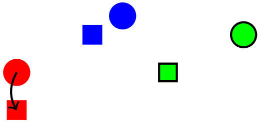


Same number?

Make a function f : Circles $\to$ Squares.

Same size?



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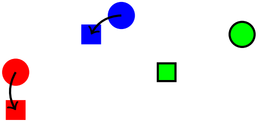
f(red circle) $=$ red square

Same size?



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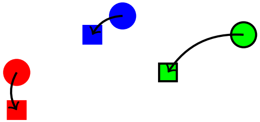

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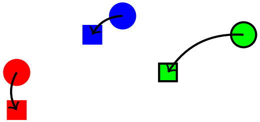

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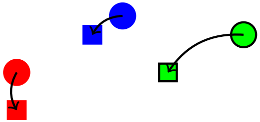
One to one.

Same size?



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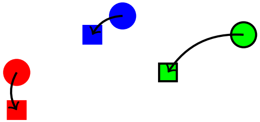
One to one. Each circle mapped to different square.

Same size?



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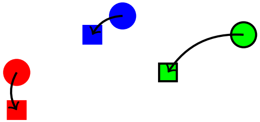

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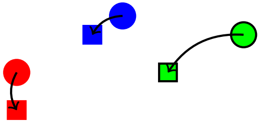

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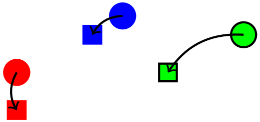

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Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

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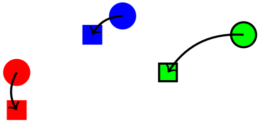

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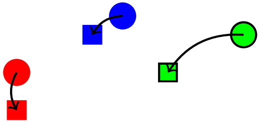

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Isomorphism principle: If there is $f : D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

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The number of subsets of a set $\{a_1, \ldots, a_n\}$.?

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Equal to the number of binary n-bit strings.

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$|P(S)|$

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$|P(S)| = |\{0,1\}^n| = 2^n$.

Countable.

How to count?

Countable.

How to count?

0,

Countable.

How to count?

0, 1,

Countable.

How to count?

0, 1, 2,

Countable.

How to count?

0, 1, 2, 3,

Countable.

How to count?

0, 1, 2, 3, ...

Countable.

How to count?

0, 1, 2, 3, . . .

The Counting numbers.

Countable.

How to count?

0, 1, 2, 3, . . .

The Counting numbers.
The natural numbers! N

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

Countable.

How to count?

0, 1, 2, 3, ...

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The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where's 0?

Which is bigger?

Where's 0?

Which is bigger?
The positive integers, Z^+, or the natural numbers, N.

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Natural numbers. 0,

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Natural numbers. $0, 1,$

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Where's 0?

More natural numbers!

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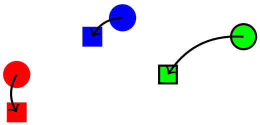

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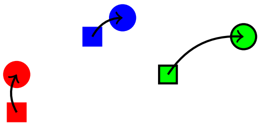

Inverse function!

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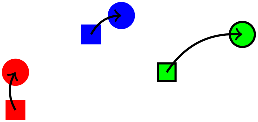

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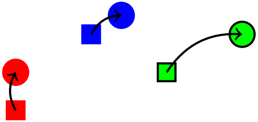

Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

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Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?

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Integers and naturals have same size!

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3	-2

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If infinite: bijection with N.

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Need to be careful.

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All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

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$B = \{0,1\}^*$.

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Can't list in "order"?

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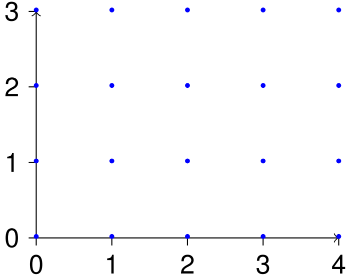

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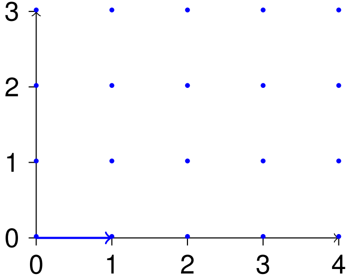

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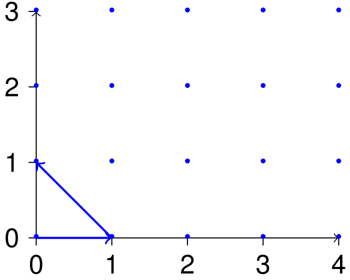

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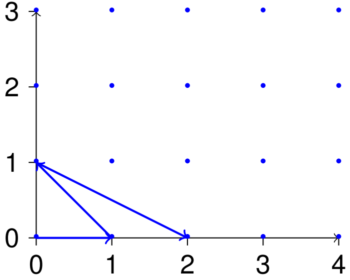

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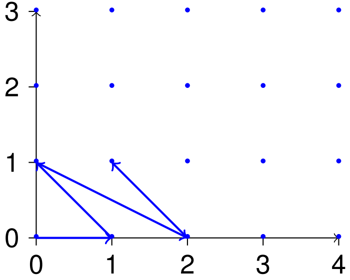

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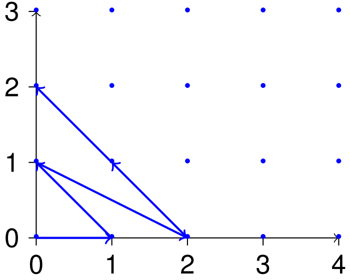

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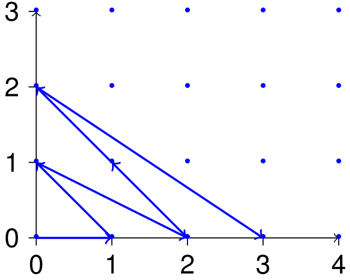

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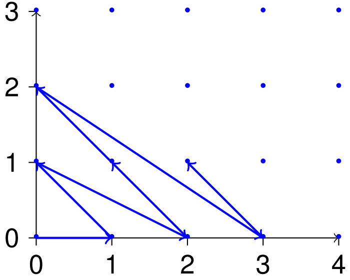

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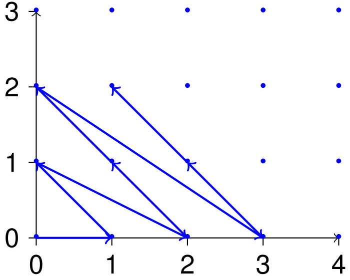

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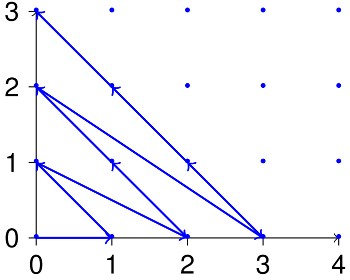

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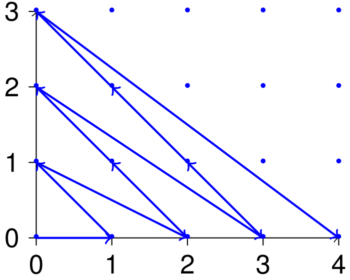

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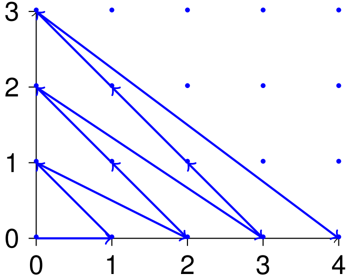

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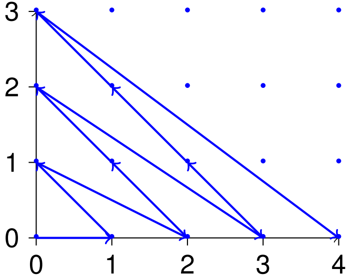

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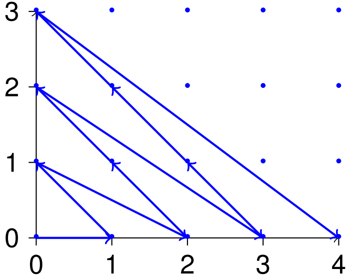

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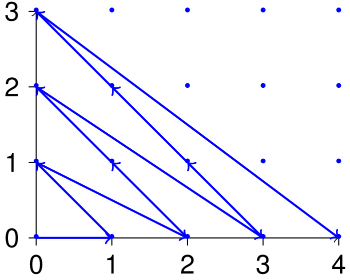

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Same size as the natural numbers!!

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Positive rational number.

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Real numbers..

Is the set of real numbers the “same size” as integers?

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Construct "diagonal" number:

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Construct “diagonal” number: .7

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Construct "diagonal" number: .77

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If reals are countable then so must $[0, 1]$.

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(The "set of all subsets of N" is the **powerset** of N.)

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$[0, 1]$ is same cardinality as nonnegative reals!

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