Counting and Probability

What's to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide.

Today: Counting!

Later this week: Probability. Professor Walrand.

Outline

1. Counting.
2. Tree
3. Rules of Counting
4. Sample with/without replacement where order does/doesn't matter.

Count?

How many outcomes possible for k coin tosses?
How many handshakes for n people?
How many 10 digit numbers?
How many 10 digit numbers without repeating digits?

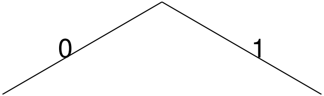
Using a tree of possibilities...

How many 3-bit strings?

How many different sequences of three bits from $\{0, 1\}$?

How would you make one sequence?

How many different ways to do that making?

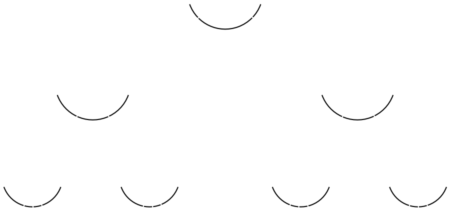


8 leaves which is $2 \times 2 \times 2$. One leaf for each string.

8 3-bit srings!

First Rule of Counting: Product Rule

Objects made by choosing from n_1, then n_2, …, then n_k
the number of objects is $n_1 \times n_2 \cdots \times n_k$.



n_1

$\times n_2$

$\times n_3$

In picture, $2 \times 2 \times 3 = 12$

Using the first rule..

How many outcomes possible for k coin tosses?

2 ways for first choice, 2 ways for second choice, ...
$2 \times 2 \cdots \times 2 = 2^k$

How many 10 digit numbers?

10 ways for first choice, 10 ways for second choice, ...
$10 \times 10 \cdots \times 10 = 10^k$

How many n digit base m numbers?

m ways for first, m ways for second, ...
m^n

Functions, polynomials.

How many functions f mapping S to T?

$|T|$ ways to choose for $f(s_1)$, $|T|$ ways to choose for $f(s_2)$, ...

....$|T|^{|S|}$

How many polynomials of degree d modulo p?

p ways to choose for first coefficient, p ways for second, ...
...p^{d+1}

p values for first point, p values for second, ...
...p^{d+1}

Permutations.

How many 10 digit numbers **without repeating a digit**?

10 ways for first, 9 ways for second, 8 ways for third, ...

... $10 * 9 * 8 \cdots * 1 = 10!$.[1]

How many different samples of size k from n numbers **without replacement.**

n ways for first choice, $n-1$ ways for second,
$n-2$ choices for third, ...

... $n * (n-1) * (n-2) \cdot * (n-k+1) = \frac{n!}{(n-k)!}$.

How many orderings of n objects are there?
Permutations of n objects.

n ways for first, $n-1$ ways for second,
$n-2$ ways for third, ...

... $n * (n-1) * (n-2) \cdot * 1 = n!$.

[1] By definition: $0! = 1$. $n! = n(n-1)(n-2)\ldots 1$.

One-to-One Functions.

How many one-to-one functions from S to S.

$|S|$ choices for $f(s_1)$, $|S|-1$ choices for $f(s_2)$, ...

So total number is $|S| \times |S|-1 \cdots 1 = |S|!$
A one-to-one function is a permutation!

Counting sets..when order doesn't matter.

How many poker hands?

$52 \times 51 \times 50 \times 49 \times 48$???

Are $A, K, Q, 10, J$ of spades
and $10, J, Q, K, A$ of spades the same?
Second Rule of Counting: If order doesn't matter count ordered objects and then divide by number of orderings.[2]

Number of orderings for a poker hand: $5!$.

Can write as...

$$\frac{52 \times 51 \times 50 \times 49 \times 48}{5!}$$

$$\frac{52!}{5! \times 47!}$$

Generic: ways to choose 5 out of 52 possibilities.

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[2] When each unordered object corresponds equal numbers of ordered objects.

When order doesn't matter.

Choose 2 out of n?

$$\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}$$

Choose 3 out of n?

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

Choose k **out of** n?

$$\frac{n!}{(n-k)! \times k!}$$

Notation: $\binom{n}{k}$ **and pronounced "**n **choose** k**."**

Simple Practice.

How many orderings of letters of CAT?
3 ways to choose first letter, 2 ways to choose second, 1 for last.
$\implies 3 \times 2 \times 1 = 3!$ orderings
How many orderings of the letters in ANAGRAM?
Ordered, except for A!
total orderings of 7 letters. 7!
total "extra counts" or orderings of two A's? 3!

Total orderings? $\frac{7!}{3!}$
How many orderings of MISSISSIPPI?
4 S's, 4 I's, 2 P's.
11 letters total!
11! ordered objects!
$4! \times 4! \times 2!$ ordered objects per "unordered object"

$\implies \frac{11!}{4!4!2!}$.

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

Order does not matter:

Second Rule: divide by number of orders – "$k!$"

$\Longrightarrow \frac{n!}{(n-k)!k!}$.

"n choose k"

With Replacement.

Order matters: $n \times n \times \ldots n = n^k$

Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!

Set: $1,2,3$ $3!$ orderings map to it.

Set: $1,2,2$ $\frac{3!}{2!}$ orderings map to it.

How do we deal with this situation?!?!

Stars and bars....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice(2^5), divide out order ???

5 dollars for Bob and 0 for Alice:
one ordered set: (B,B,B,B,B).

4 for Bob and 1 for Alice:
5 ordered sets: (A,B,B,B,B) ; (B,A,B,B,B); ...

Well, we can list the possibilities.
$0+5$, $1+4$,$2+3$, $3+2$, $4+1$, $5+0$.

For 2 numbers adding to k, we get $k+1$.

For 3 numbers adding to k?

Stars and Bars.

How many ways to add up n numbers to equal k?

Or: k choices from set of n possibilities with replacement.
Sample with replacement where order just doesn't matter.

How many ways can Alice, Bob, and Eve split 5 dollars.

Think of Five dollars as Five stars: $\star\star\star\star\star$.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: $\star\star|\star|\star\star$.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: $|\star|\star\star\star\star$.

Each split $\implies$ a sequence of stars and bars.
Each sequence of stars and bars $\implies$ a split.

Counting Rule: if there is a one-to-one mapping between two sets they have the same size!

Stars and Bars.

How many different 5 star and 2 bar diagrams?

7 positions in which to place the 2 bars.

$\binom{7}{2}$ ways to do so and $\binom{7}{2}$ ways to split 5$ among 3 people.

Ways to add up n numbers to sum to k? or

" k from n with replacement where order doesn't matter."

In general, k stars $n-1$ bars.

$$\star\star|\star|\cdots|\star\star.$$

$n+k-1$ positions from which to choose $n-1$ bar positions.

$$\binom{n+k-1}{n-1}$$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$

Inclusion/Exclusion Rule: For any S and T,
$|S \cup T| = |S| + |T| - |S \cap T|$.

Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit.$|S| = 10^9$

T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

k Samples with replacement from n items: n^k.
Sample without replacement: $\frac{n!}{(n-k)!}$

Second rule: when order doesn't matter divide..when possible.

Sample without replacement and order doesn't matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
"n choose k"

One-to-one rule: equal in number if one-to-one correspondence.

Sample with replacement and order doesn't matter: $\binom{k+n-1}{n}$.

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Inclusion/Exclusion Rule: For any S and T,
$|S \cup T| = |S| + |T| - |S \cap T|$.