CS70: Jean Walrand: Lecture 20.

Modeling Uncertainty: Probability Space

1. Key Points
2. Random Experiments
3. Probability Space

Key Points

- Uncertainty does not mean “nothing is known”
- How to best make decisions under uncertainty?
  - Buy stocks
  - Detect signals (transmitted bits, speech, images, radar, diseases, etc.)
  - Control systems (Internet, airplane, robots, self-driving cars, schedule surgeries in a hospital, etc.)
- How to best use ‘artificial’ uncertainty?
  - Play games of chance
  - Design randomized algorithms.
- Probability
  - Models knowledge about uncertainty
  - Discovers best way to use that knowledge in making decisions

The Magic of Probability

Uncertainty: vague, fuzzy, confusing, scary, hard to think about.

Probability: A precise, unambiguous, simple(!) way to think about uncertainty.

Uncertainty = Fear

Probability = Serenity

Our mission: help you discover the serenity of Probability, i.e., enable you to think clearly about uncertainty.

Your cost: focused attention and practice on examples and problems.

Random Experiment: Flip one Fair Coin

Flip a fair coin: (*One flips or tosses a coin*)

- Possible outcomes: Heads (H) and Tails (T) (*One flip yields either ‘heads’ or ‘tails’.*)
- Likelihoods: $H : 50\%$ and $T : 50\%$

Random Experiment: Flip one Fair Coin

Flip a fair coin:

What do we mean by the likelihood of tails is 50%?

Two interpretations:

- Single coin flip: 50% chance of ‘tails’ [subjectivist]

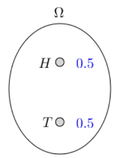
  Willingness to bet on the outcome of a single flip
- Many coin flips: About half yield ‘tails’ [frequentist]

  Makes sense for many flips
- Question: Why does the fraction of tails converge to the same value every time? Statistical Regularity! Deep!

Random Experiment: Flip one Fair Coin

Flip a fair coin: model

Physical Experiment



Probability Model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set Ω of outcomes: $\Omega = \{H, T\}$.
  - A probability assigned to each outcome: $Pr[H] = 0.5, Pr[T] = 0.5$.

Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

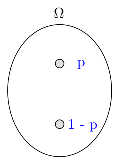
- Possible outcomes: Heads (H) and Tails (T)
- Likelihoods: $H : p \in (0,1)$ and $T : 1-p$
- Frequentist Interpretation:

  Flip many times $\Rightarrow$ Fraction $1-p$ of tails
- Question: How can one figure out p? Flip many times
- Tautolgy? No: Statistical regularity!

Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin: model

Physical Experiment



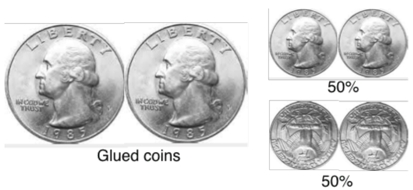
Probability Model

Flip Two Fair Coins

- Possible outcomes: $\{HH, HT, TH, TT\} \equiv \{H,T\}^2$.
- Note: $A \times B := \{(a,b) \mid a \in A, b \in B\}$ and $A^2 := A \times A$.
- Likelihoods: $1/4$ each.

Flip Glued Coins

Flips two coins glued together side by side:



- Possible outcomes: $\{HH, TT\}$.
- Likelihoods: $HH : 0.5, TT : 0.5$.
- Note: Coins are glued so that they show the same face.

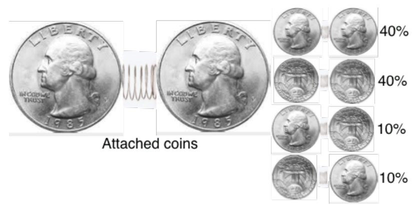
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: $\{HT, TH\}$.
- Likelihoods: $HT : 0.5, TH : 0.5$.
- Note: Coins are glued so that they show different faces.

Flip two Attached Coins

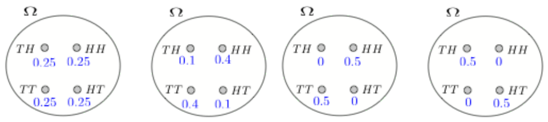
Flips two coins attached by a spring:



- Possible outcomes: $\{HH, HT, TH, TT\}$.
- Likelihoods: $HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4$.
- Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.

Flipping Two Coins

Here is a way to summarize the four random experiments:



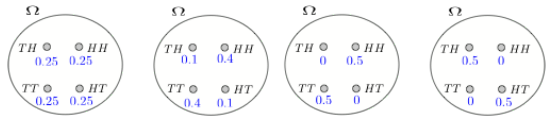
[1] [2] [3] [4]

- Ω is the set of *possible* outcomes;
- Each outcome has a probability (likelihood);
- The probabilities are ≥ 0 and add up to 1;
- Fair coins: [1]; Glued coins: [3], [4];

  Spring-attached coins: [2];

Flipping Two Coins

Here is a way to summarize the four random experiments:



[1] [2] [3] [4]

Important remarks:

- Each outcome describes the two coins.
- E.g., HT is one outcome of the experiment.
- It is wrong to think that the outcomes are $\{H,T\}$ and that one picks twice from that set.
- Indeed, this viewpoint misses the relationship between the two flips.
- Each $\omega \in \Omega$ describes one outcome of the complete experiment.
- Ω and the probabilities specify the random experiment.

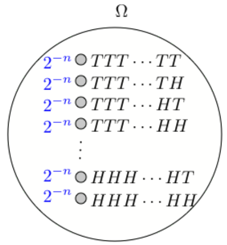
Flipping n times

Flip a fair coin n times (some $n \geq 1$):

- Possible outcomes: $\{TT\cdots T, TT\cdots H, \ldots, HH\cdots H\}$.

  Thus, 2^n possible outcomes.
- Note: $\{TT\cdots T, TT\cdots H, \ldots, HH\cdots H\} = \{H,T\}^n$.

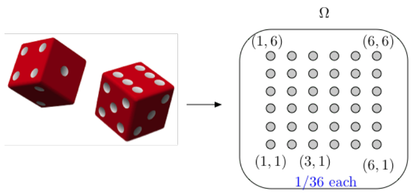
  $A^n := \{(a_1,\ldots,a_n) \mid a_1 \in A, \ldots, a_n \in A\}$. $|A^n| = |A|^n$.
- Likelihoods: $1/2^n$ each.



Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes:
  $\{1,2,3,4,5,6\}^2 = \{(a,b) \mid 1 \leq a,b \leq 6\}$.
- Likelihoods: $1/36$ for each.



Physical Experiment   Probability Model

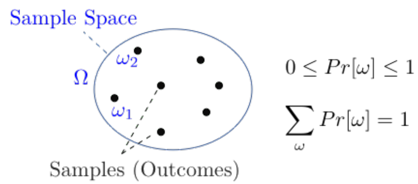
Probability Space.

1. A "random experiment":
   (a) Flip a biased coin;
   (b) Flip two fair coins;
   (c) Deal a poker hand.
2. A set of possible outcomes: Ω.
   (a) $\Omega = \{H,T\}$;
   (b) $\Omega = \{HH, HT, TH, TT\}$; $|\Omega| = 4$;
   (c) $\Omega = \{\underline{A♠\ A♢\ A♣\ A♡\ K♠}, \underline{A♠\ A♢\ A♣\ A♡\ Q♠}, \ldots\}$
   $|\Omega| = \binom{52}{5}$.
3. Assign a probability to each outcome: $Pr : \Omega \to [0,1]$.
   (a) $Pr[H] = p, Pr[T] = 1-p$ for some $p \in [0,1]$
   (b) $Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$
   (c) $Pr[\,\underline{A♠\ A♢\ A♣\ A♡\ K♠}\,] = \cdots = 1/\binom{52}{5}$

Probability Space: formalism.

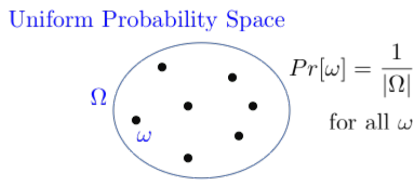
Ω is the **sample space.**
$\omega \in \Omega$ is a **sample point**. (Also called an **outcome**.)
Sample point ω has a probability $Pr[\omega]$ where

- $0 \leq Pr[\omega] \leq 1$;
- $\sum_{\omega \in \Omega} Pr[\omega] = 1$.



Probability Space: Formalism.

In a **uniform probability space** each outcome ω is equally probable: $Pr[\omega] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$.

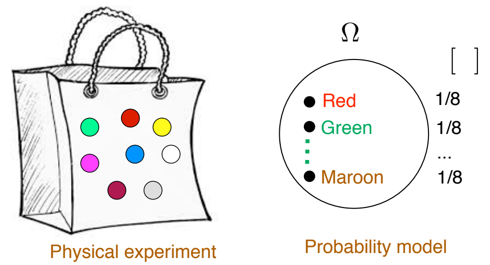


Examples:

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
- Flipping a biased coin is not a uniform probability space.

Probability Space: Formalism

Simplest physical model of a uniform probability space:



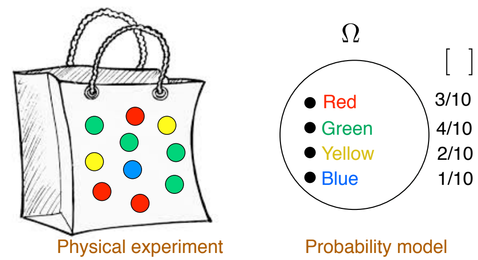
A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

$$\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}$$

$$Pr[\text{blue}] = \frac{1}{8}.$$

Probability Space: Formalism

Simplest physical model of a non-uniform probability space:



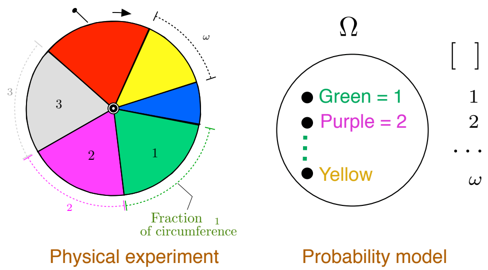
$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] = \frac{4}{10}, \text{ etc.}$$

Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

Probability Space: Formalism

Physical model of a general non-uniform probability space:



The roulette wheel stops in sector ω with probability p_ω.

$$\Omega = \{1, 2, 3, \ldots, N\}, Pr[\omega] = p_\omega.$$

An important remark

- The random experiment selects one and only one outcome in Ω.
- For instance, when we flip a fair coin twice
  - $\Omega = \{HH, TH, HT, TT\}$
  - The experiment selects *one* of the elements of Ω.
- In this case, its would be wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.
- Why? Because this would not describe how the two coin flips are related to each other.
- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets HH or TT with probability 50% each. This is not captured by 'picking two outcomes.'

Lecture 15: Summary

Modeling Uncertainty: Probability Space

1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0,1]; \sum_\omega Pr[\omega] = 1$.
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.