CS70: Jean Walrand: Lecture 21.

Events, Conditional Probability

1. Probability Basics Review
2. Events
3. Conditional Probability

Probability Basics Review

Setup:

- Random Experiment.
  Flip a fair coin twice.
- Probability Space.
  - **Sample Space:** Set of outcomes, Ω.
    $\Omega = \{HH, HT, TH, TT\}$
    (Note: Not $\Omega = \{H, T\}$ with two picks!)
  - **Probability:** $Pr[\omega]$ for all $\omega \in \Omega$.
    $Pr[HH] = \cdots = Pr[TT] = 1/4$
    1. $0 \leq Pr[\omega] \leq 1$.
    2. $\sum_{\omega \in \Omega} Pr[\omega] = 1$.

Set notation review

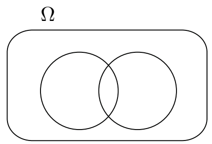


Figure: Two events

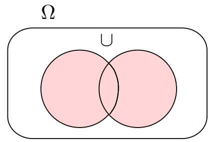


Figure: Union (or)

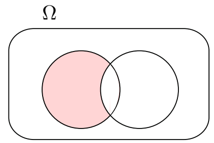


Figure: Difference (A, not B)

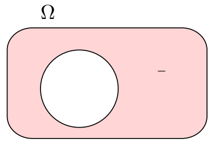


Figure: Complement (not)

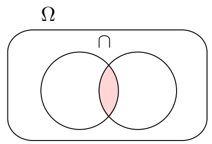


Figure: Intersection (and)

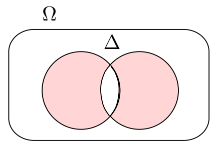


Figure: Symmetric difference (only one)

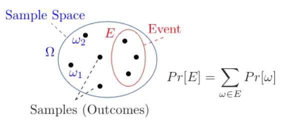
Probability of exactly one ‘heads’ in two coin flips?

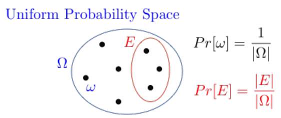
Idea: Sum the probabilities of all the different outcomes that have exactly one ‘heads’: HT, TH.

This leads to a definition!

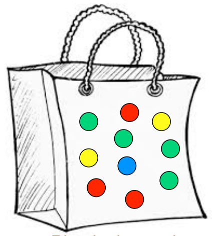
Definition:

- An **event,** E**,** is a subset of outcomes: $E \subset \Omega$.
- The **probability of** E is defined as $Pr[E] = \sum_{\omega \in E} Pr[\omega]$.

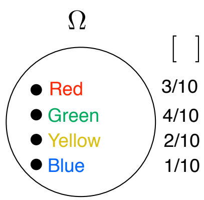




Event: Example



Physical experiment



Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$
$$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] = \frac{4}{10}, \text{ etc.}$$

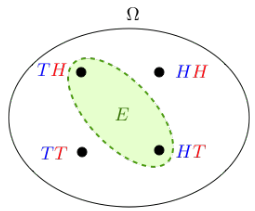
$$E = \{Red, Green\} \Rightarrow Pr[E] = \frac{3+4}{10} = \frac{3}{10} + \frac{4}{10} = Pr[\text{Red}] + Pr[\text{Green}].$$

Probability of exactly one heads in two coin flips?

Sample Space, $\Omega = \{HH, HT, TH, TT\}$.

Uniform probability space:
$Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$.

Event, E, “exactly one heads”: $\{TH, HT\}$.



$$Pr[E] = \sum_{\omega \in E} Pr[\omega] = \frac{|E|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}.$$

Example: 20 coin tosses.

20 coin tosses

Sample space: $\Omega =$ set of 20 fair coin tosses.
$\Omega = \{T, H\}^{20} \equiv \{0,1\}^{20}$; $|\Omega| = 2^{20}$.

- What is more likely?
  - $\omega_1 := (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$, or
  - $\omega_2 := (1,0,1,1,0,0,0,1,0,1,0,1,1,0,1,1,1,0,0,0)$?

  Answer: Both are equally likely: $Pr[\omega_1] = Pr[\omega_2] = \frac{1}{|\Omega|}$.
- What is more likely?
  (E_1) Twenty Hs out of twenty, or
  (E_2) Ten Hs out of twenty?

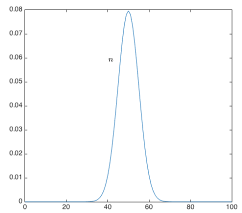
  Answer: Ten Hs out of twenty.

  Why? There are many sequences of 20 tosses with ten Hs; only one with twenty Hs. $\Rightarrow Pr[E_1] = \frac{1}{|\Omega|} \ll Pr[E_2] = \frac{|E_2|}{|\Omega|}$.

$$|E_2| = \binom{20}{10} = 184,756.$$

Probability of n heads in 100 coin tosses.

$\Omega = \{H, T\}^{100}$; $|\Omega| = 2^{100}$.



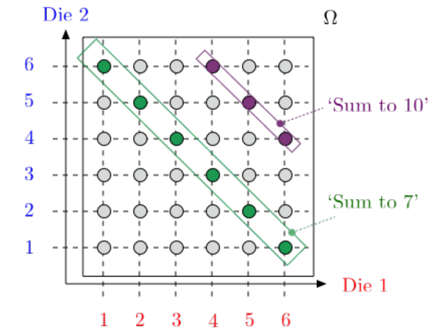
Event E_n = 'n heads'; $|E_n| = \binom{100}{n}$

$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$

Observe:

- Concentration around mean: Law of Large Numbers;
- Bell-shape: Central Limit Theorem.

Roll a red and a blue die.



$Pr[\text{Sum to 7}] = \frac{6}{36}$ $\quad$ $Pr[\text{Sum to 10}] = \frac{3}{36}$

Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$.
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ "100 coin tosses with exactly 50 heads"

$|E|$?
Choose 50 positions out of 100 to be heads.
$|E| = \binom{100}{50}$.

$$Pr[E] = \frac{\binom{100}{50}}{2^{100}}.$$

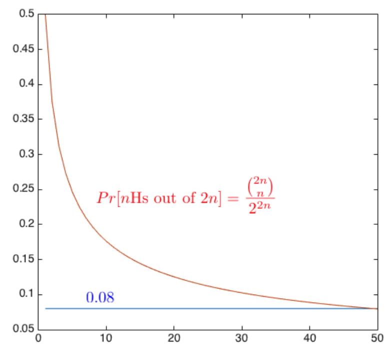
Calculation.
Stirling formula (for large n):

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

$$\binom{2n}{n} \approx \frac{\sqrt{4\pi n}(2n/e)^{2n}}{[\sqrt{2\pi n}(n/e)^n]^2} \approx \frac{4^n}{\sqrt{\pi n}}.$$

$$Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{2^{2n}} = \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{50\pi}} \approx .08.$$

Exactly 50 heads in 100 coin tosses.



Probability is Additive

Theorem

(a) If events A and B are disjoint, i.e., $A \cap B = \emptyset$, then

$$Pr[A \cup B] = Pr[A] + Pr[B].$$

(b) If events $A_1, \ldots, A_n$ are pairwise disjoint,
i.e., $A_k \cap A_m = \emptyset, \forall k \neq m$, then

$$Pr[A_1 \cup \cdots \cup A_n] = Pr[A_1] + \cdots + Pr[A_n].$$

Proof:

Obvious.

Consequences of Additivity

Theorem

(a) $Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$;
(inclusion-exclusion property)

(b) $Pr[A_1 \cup \cdots \cup A_n] \leq Pr[A_1] + \cdots + Pr[A_n]$;
(union bound)

(c) If $A_1, \ldots A_N$ are a partition of Ω, i.e.,
pairwise disjoint and $\cup_{m=1}^{N} A_m = \Omega$, then

$$Pr[B] = Pr[B \cap A_1] + \cdots + Pr[B \cap A_N].$$

(law of total probability)

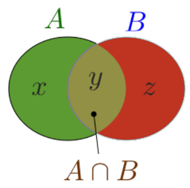
Proof:

(b) is obvious.

See next two slides for (a) and (c).

Inclusion/Exclusion

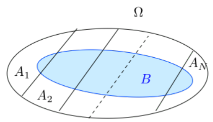
$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$



$$Pr[A] = x + y$$
$$Pr[B] = y + z$$
$$Pr[A \cap B] = y$$
$$Pr[A \cup B] = x + y + z$$

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.

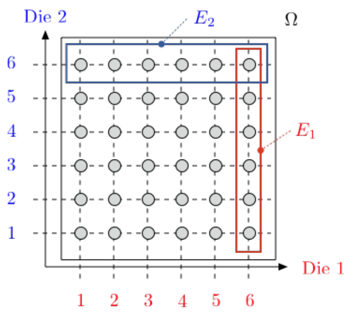


Then,

$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$

Indeed, B is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$.

Roll a Red and a Blue Die.

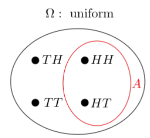


$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

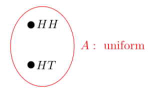
E_1 = 'Red die shows 6'; E_2 = 'Blue die shows 6'
$E_1 \cup E_2$ = 'At least one die shows 6'
$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}$.

Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads?
$\Omega = \{HH, HT, TH, TT\}$; Uniform probability space.
Event A = first flip is heads: $A = \{HH, HT\}$.



New sample space: A; uniform still.



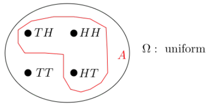
Event B = two heads.

The probability of two heads if the first flip is heads.
The probability of B given A is $1/2$.

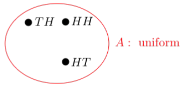
A similar example.

Two coin flips. At least one of the flips is heads.
→ Probability of two heads?

$\Omega = \{HH, HT, TH, TT\}$; uniform.
Event $A =$ at least one flip is heads. $A = \{HH, HT, TH\}$.



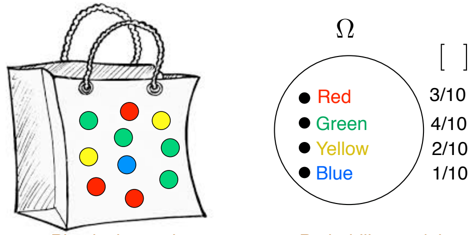
New sample space: A; uniform still.



Event $B =$ two heads.

The probability of two heads if at least one flip is heads.
The probability of B **given** A is $1/3$.

Conditional Probability: A non-uniform example



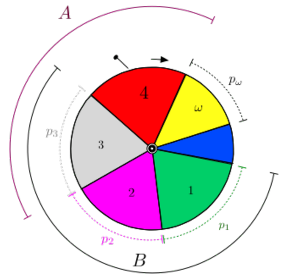
Physical experiment   Probability model

$\Omega = \{\text{Red, Green, Yellow, Blue}\}$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]}$$

Another non-uniform example

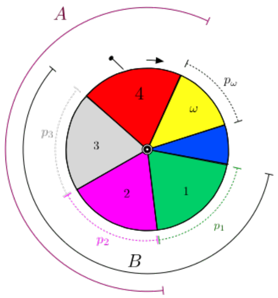
Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.
Let $A = \{3, 4\}, B = \{1, 2, 3\}$.



$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

Yet another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.
Let $A = \{2, 3, 4\}, B = \{1, 2, 3\}$.

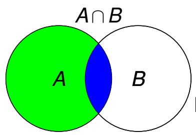


$$Pr[A|B] = \frac{p_2 + p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

Conditional Probability.

Definition: The **conditional probability** of B given A is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$



In A!
In B?
Must be in $A \cap B$.

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}.$$

Summary

Events, Conditional Probability

Key Ideas:

- Conditional Probability:
  $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$
- All these are possible:
  $Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A]$.