CS70: Jean Walrand: Lecture 23.

Bayes' Rule, Independence, Mutual Independence

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1. Conditional Probability: Review
2. Bayes' Rule: Another Look
3. Independence
4. Mutual Independence

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Conditional Probability: Pictures

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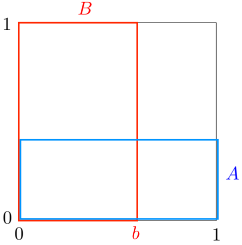
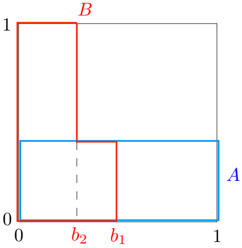
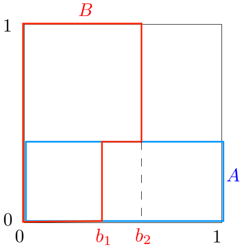
Illustrations: Pick a point uniformly in the unit square







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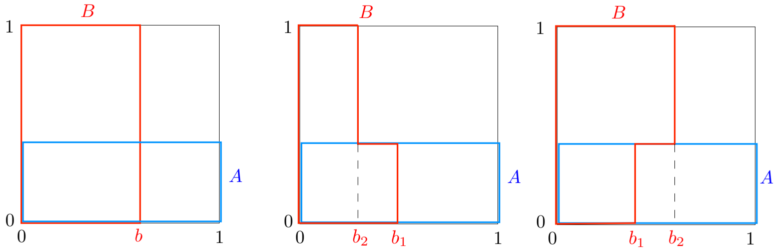

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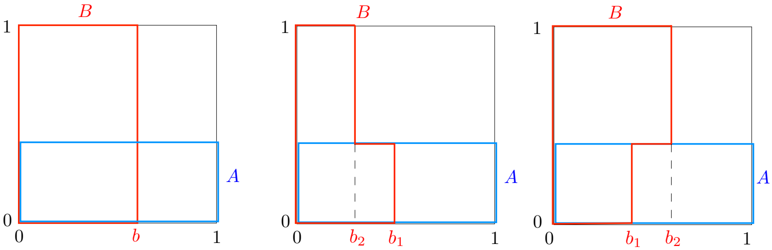

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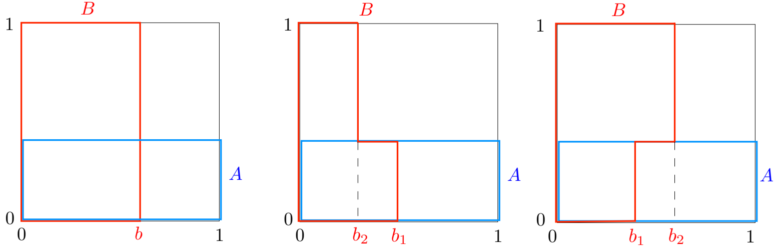

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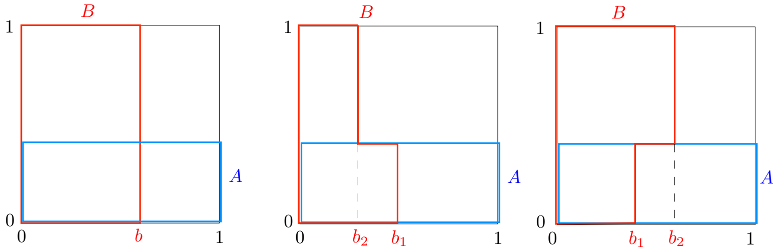

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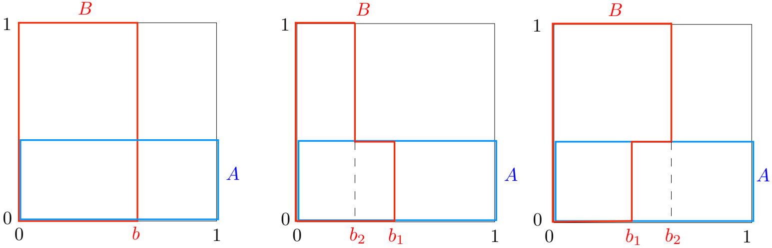

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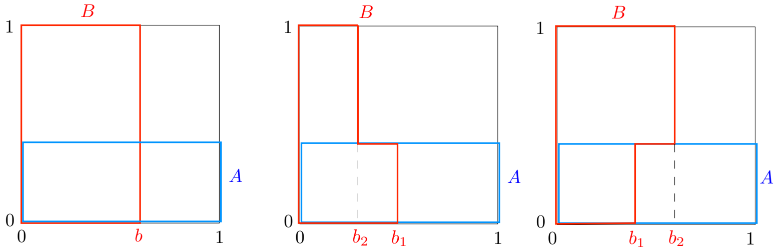

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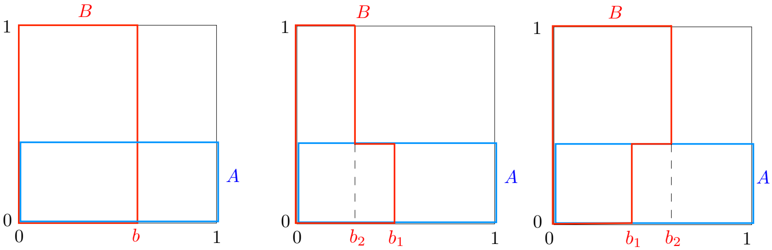

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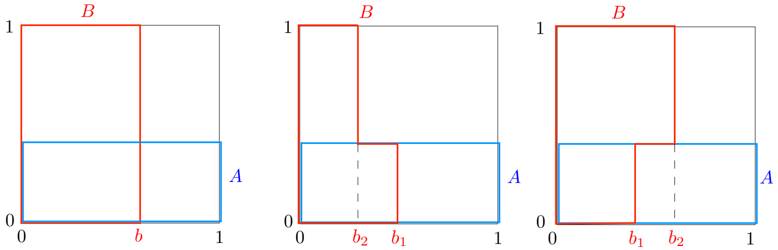

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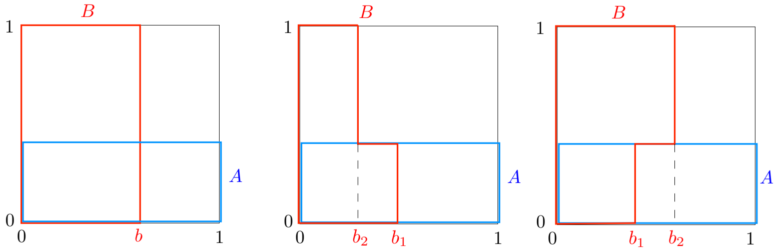

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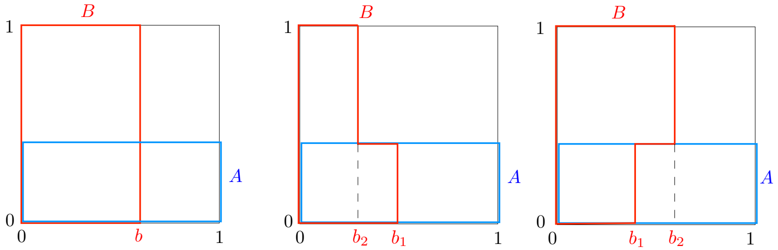

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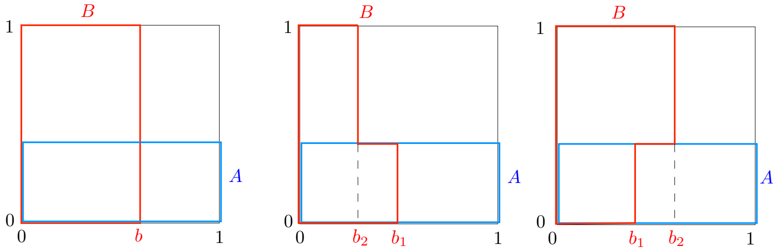

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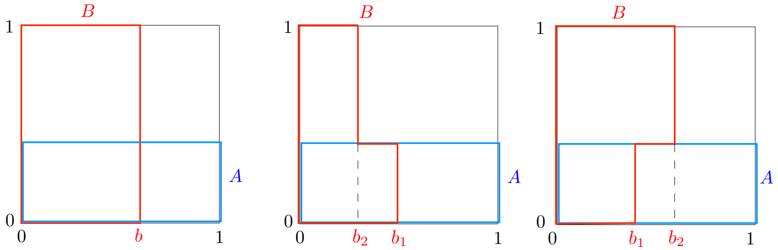

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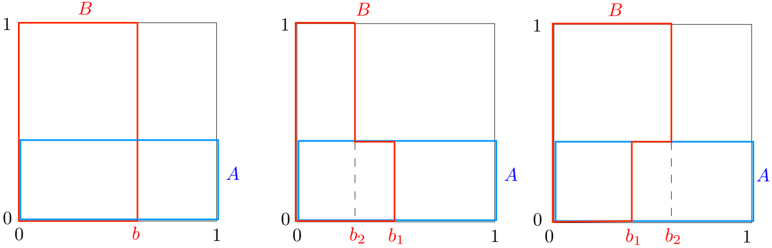

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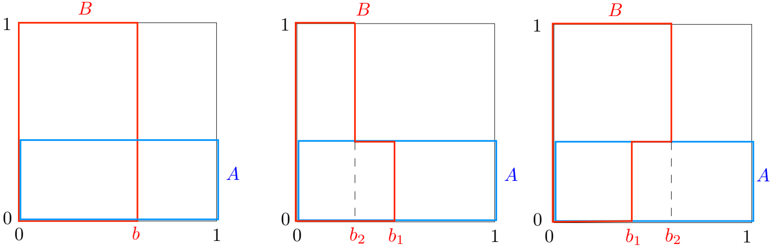

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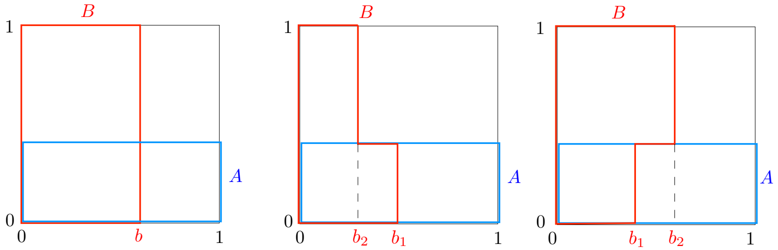

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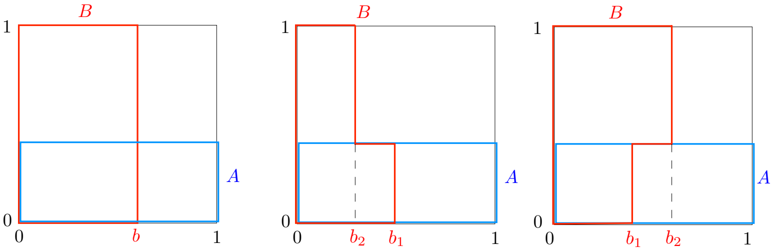

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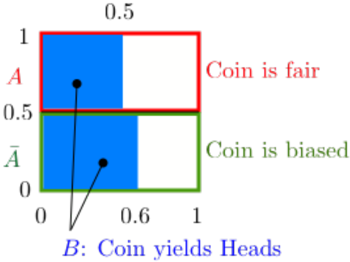

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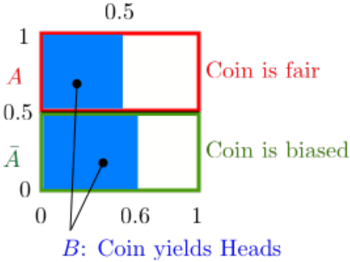

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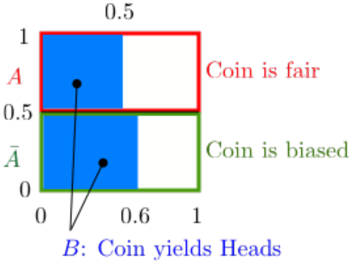

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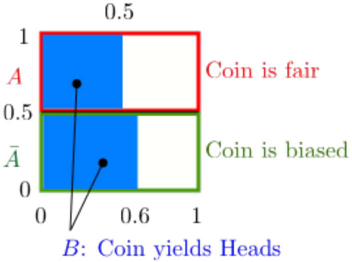

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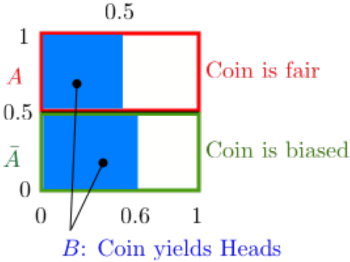

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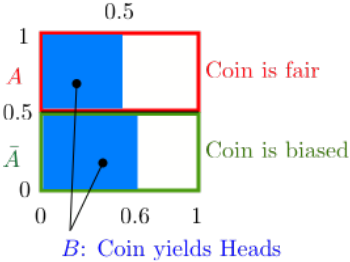

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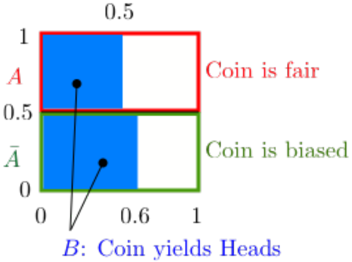

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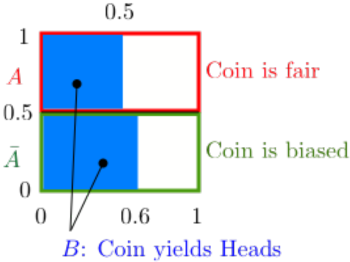

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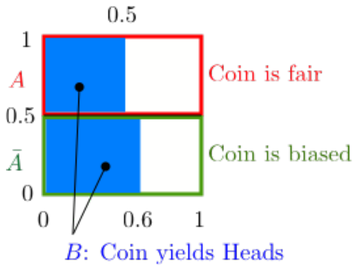

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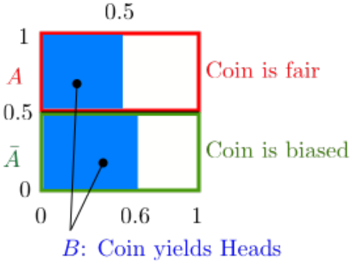

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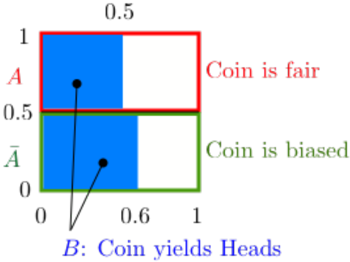

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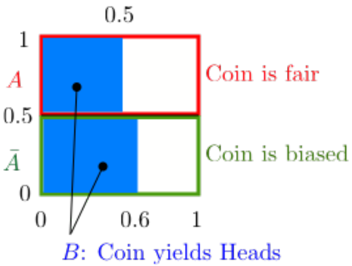

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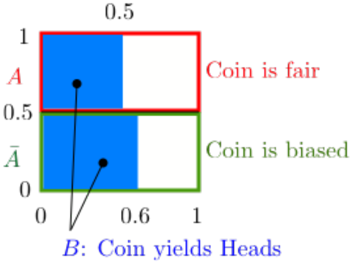

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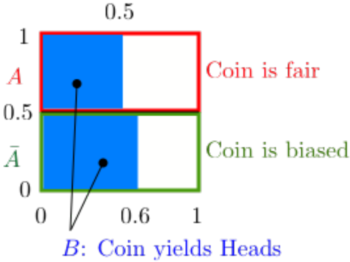

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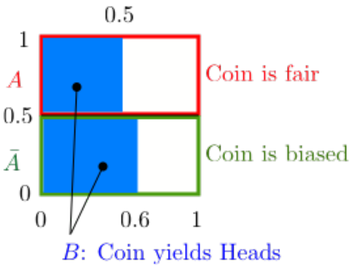

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$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$

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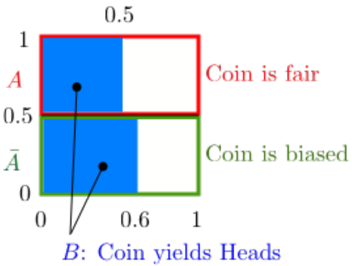

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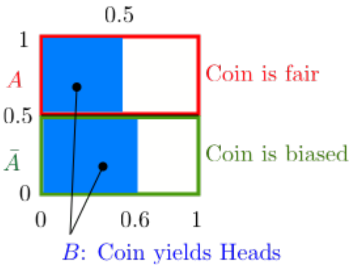
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6}$$

Bayes and Biased Coin



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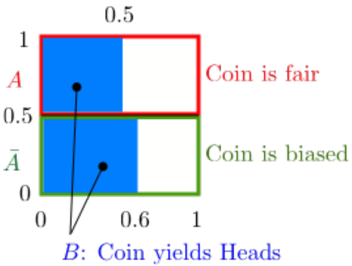

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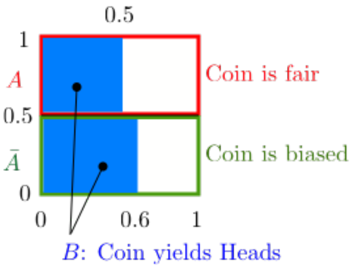

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Bayes: General Case

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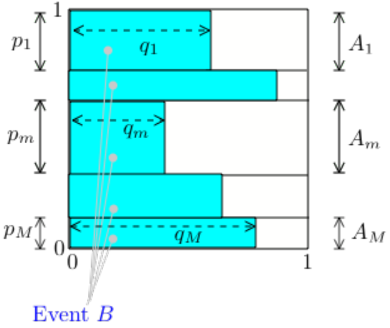

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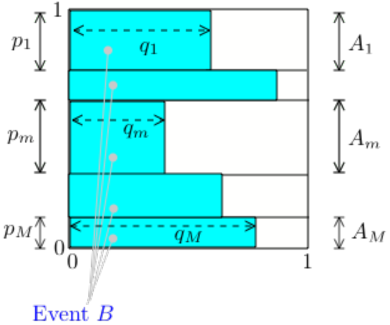

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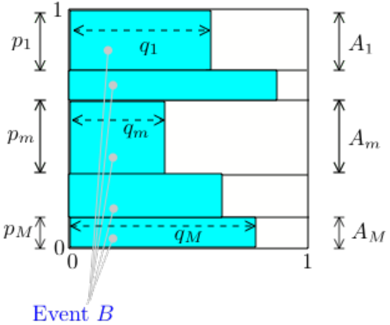

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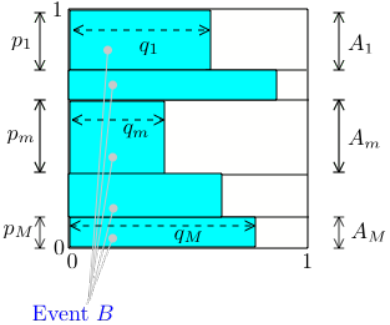

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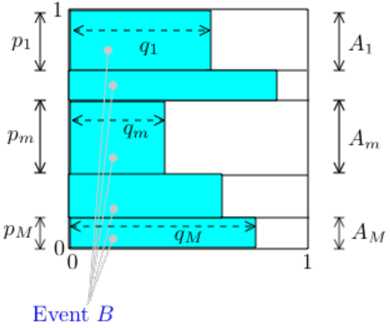

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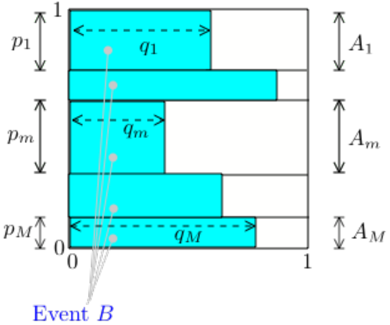

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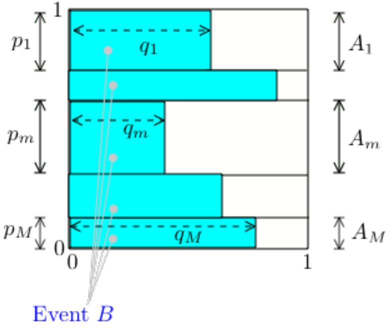

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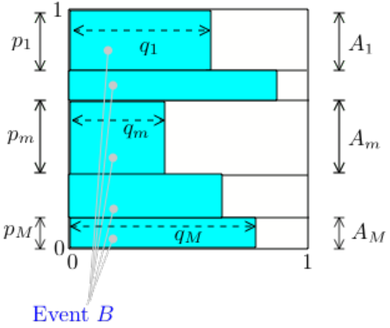

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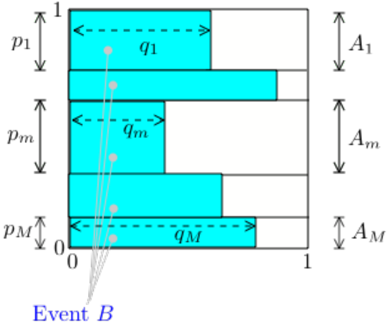

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Bayes Rule

Another picture:

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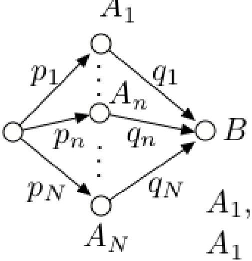

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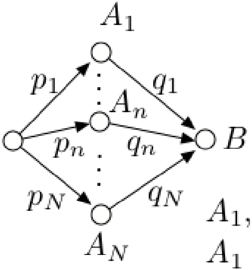

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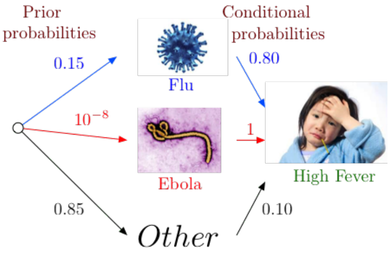

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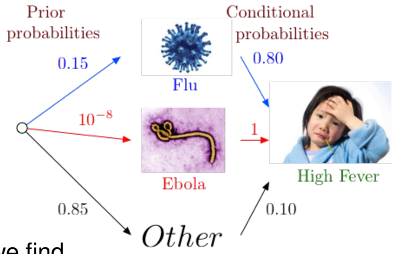

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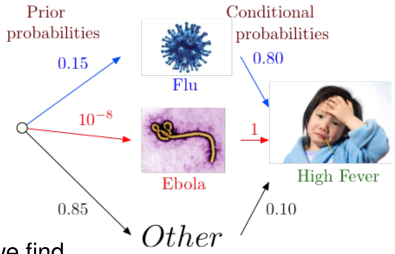

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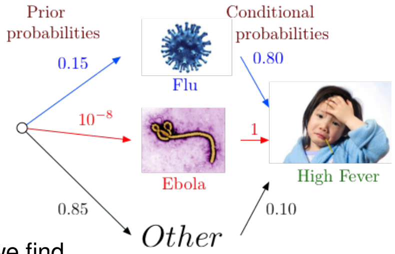

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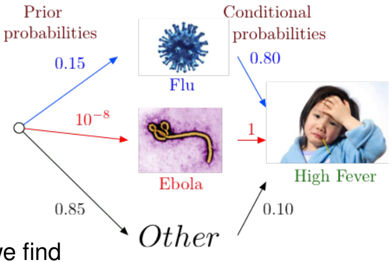

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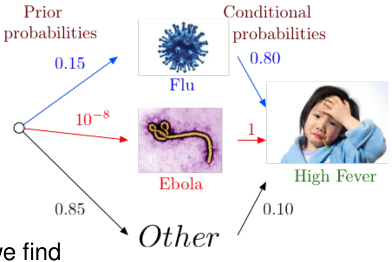

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The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

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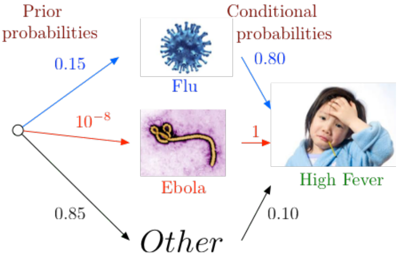

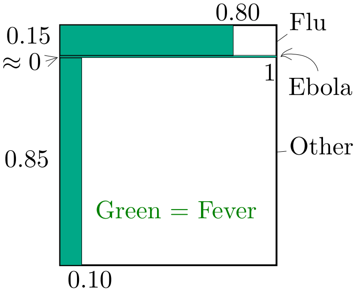

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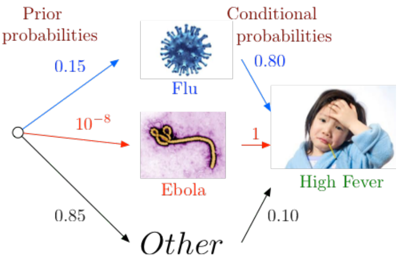

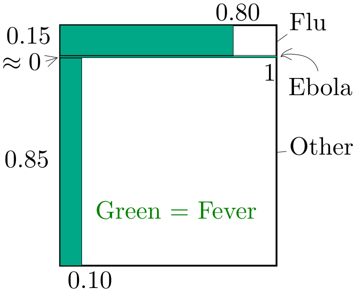

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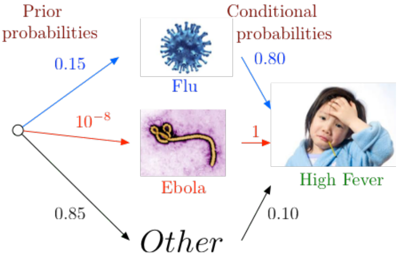

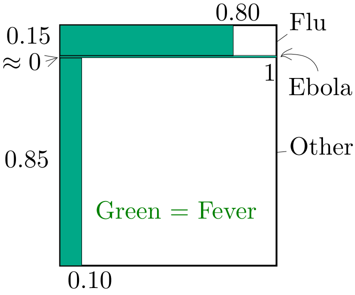

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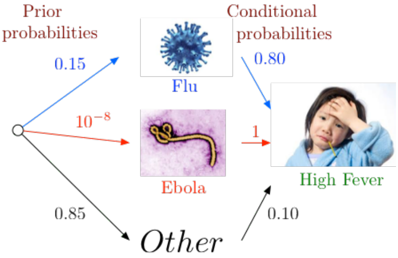

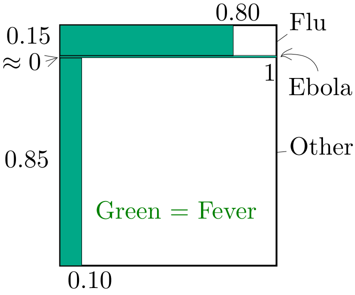

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This example shows the importance of the prior probabilities.

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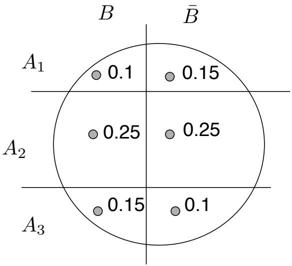

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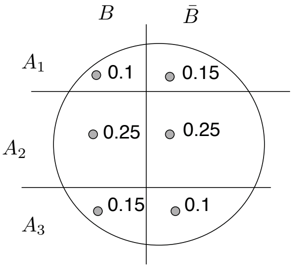

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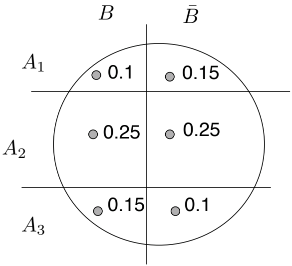

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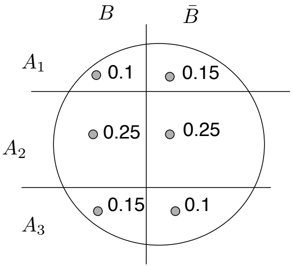

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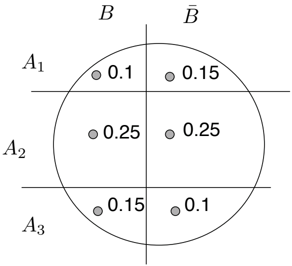

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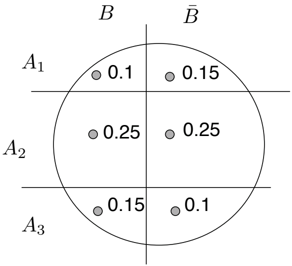

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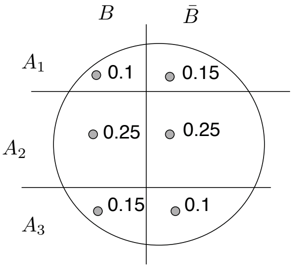

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Flip two fair coins. Let

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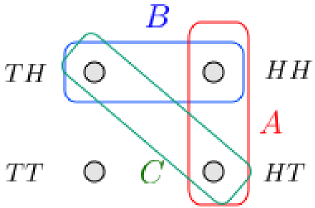

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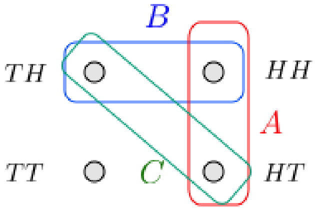

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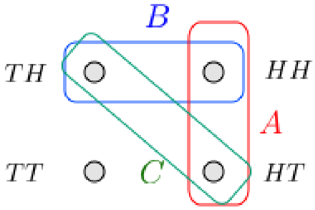

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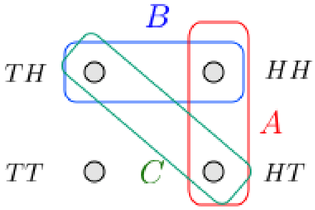

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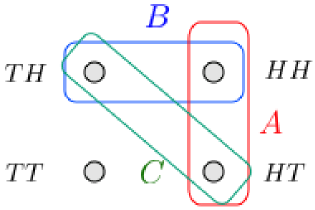

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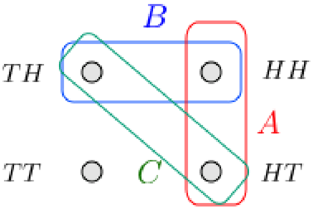

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Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

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For instance, the fact that there are more heads than tails in the first five flips of a coin is independent of the fact there are fewer heads than tails in flips $6, \ldots, 13$.

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Proof:

We show that
$Pr[A \cap B^c \cap C \cap \cdots \cap G^c \cap H] = Pr[A]Pr[B^c] \cdots Pr[G^c]Pr[H]$.

Assume that this is true when there are at most n complements.

Base case: $n = 0$ true by definition of mutual independence.

Induction step: Assume true for n. Check for $n+1$:

$A \cap B^c \cap C \cap \cdots \cap G^c \cap H =$
$A \cap B^c \cap C \cap \cdots \cap F \cap H \setminus A \cap B^c \cap C \cap \cdots \cap G \cap H$. Hence,

$$
\begin{aligned}
& Pr[A \cap B^c \cap C \cap \cdots \cap G^c \cap H] \\
& \quad = Pr[A \cap B^c \cap C \cap \cdots \cap F \cap H] - Pr[A \cap B^c \cap C \cap \cdots \cap G \cap H] \\
& \quad = Pr[A]Pr[B^c] \cdots Pr[F]Pr[H] - Pr[A]Pr[B^c] \cdots Pr[F]Pr[G]Pr[H] \\
& \quad = Pr[A]Pr[B^c] \cdots Pr[F]Pr[H](1 - Pr[G]) \\
& \quad = Pr[A]Pr[B^c] \cdots Pr[F]Pr[G^c]Pr[H]. \quad \square
\end{aligned}
$$

Summary.

Bayes' Rule, Independence, Mutual Independence

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Main results:

- Bayes' Rule: $Pr[A_m|B] = p_m q_m / (p_1 q_1 + \cdots + p_M q_M)$.

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Main results:

- Bayes' Rule: $Pr[A_m|B] = p_m q_m/(p_1 q_1 + \cdots + p_M q_M)$.
- Mutual Independence: Events defined by disjoint collections of mutually independent events are mutually independent.