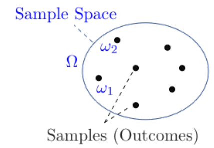
CS70: Sanjit Seshia & Jean Walrand: Lecture 24b.

Review M2 - Probability

Probability: Midterm 2 Review.

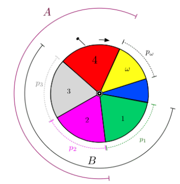
- Framework:
  - Probability Space
  - Conditional Probability & Bayes' Rule
  - Independence
  - Mutual Independence
- Notes:
  - Note 25b: Page 1 + Bayes' Rule on page 2.
  - Note 13
  - Note 14

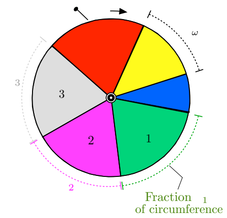
Review: Probability Space



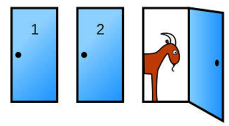
$0 \leq Pr[\omega] \leq 1$

$\sum_{\omega} Pr[\omega] = 1$



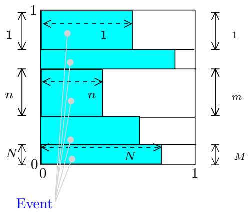


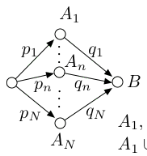
$Pr[A|B] = Pr[A \cap B]/Pr[B]$.
$Pr[A \cap B \cap C]$
$= Pr[A]Pr[B|A]Pr[C|A \cap B]$.



Review: Bayes' Rule

- Priors: $Pr[A_n] = p_n, n = 1, \ldots, M$
- Conditional Probabilities: $Pr[B|A_n] = q_n, n = 1, \ldots, N$
- $\Rightarrow$ Posteriors: $Pr[A_n|B] = \frac{p_n q_n}{p_1 q_1 + \cdots + p_N q_N}$





$p_n = Pr[A_n]$
$q_n = Pr[B|A_n]$

$A_1, \ldots, A_N$ disjoint
$A_1 \cup \cdots \cup A_N = \Omega$

Bayes' Rule: Examples

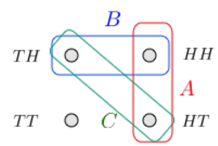
Let $p'_n = Pr[A_n|B]$ be the posterior probabilities.

Thus, $p'_n = p_n q_n / (p_1 q_1 + \cdots + p_N q_n)$.

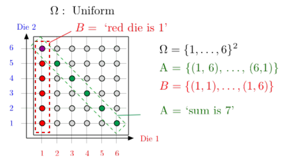
Questions: Is it true that

- if $q_n > q_k$, then $p'_n > p'_k$? Not necessarily.
- if $p_n > p_k$, then $p'_n > p'_k$? Not necessarily.
- if $p_n > p_k$ and $q_n > q_k$, then $p'_n > p'_k$? Yes.
- if $q_n = 1$, then $p'_n > 0$? Not necessarily.
- if $p_n = 1/N$ for all n, then MLE = MAP? Yes.

Review: Independence



Pairwise, but not mutually



"First coin yields 1" and "Sum is 7" are independent

If $\{A_j, i \in J\}$ are mutually independent, then $[A_1 \cap \bar{A}_2] \Delta A_3$ and $A_4 \setminus A_5$ are independent.

Our intuitive meaning of "independent events" is mutual independence.

Review: Independence

Recall

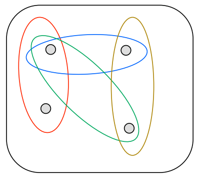
- A and B are independent if $Pr[A \cap B] = Pr[A]Pr[B]$.
- $\{A_j, j \in J\}$ are mutually independent if
  $Pr[\cap_{j \in K} A_j] = \Pi_{j \in K} Pr[A_j], \forall$ finite $K \subset J$.

Thus, A, B, C, D are mutually independent if there are

- independent 2 by 2:
  $Pr[A \cap B] = Pr[A]Pr[B], \ldots, Pr[C \cap D] = Pr[C]Pr[D]$
- by 3: $Pr[A \cap B \cap C] = Pr[A]Pr[B]Pr[C], \ldots, Pr[B \cap C \cap D] = Pr[B]Pr[C]Pr[D]$
- by 4: $Pr[A \cap B \cap C \cap D] = Pr[A]Pr[B]Pr[C]Pr[D]$.

Independence: Question 1

Consider the uniform probability space and the events A, B, C, D.



Which maximal collections of events among A, B, C, D are pairwise independent?

$\{A, B, C\}$, and $\{B, C, D\}$

Can you find three events among A, B, C, D that are mutually independent?

No: We would need an outcome with probability $1/8$.

Independence: Question 2

Let $\Omega = \{1, 2, \ldots, p\}$ be a uniform probability space where p is prime.

Can you find two independent events A and B with $Pr[A], Pr[B] \in (0, 1)$?

Let $a = |A|, b = |B|, c = |A \cap B|$.

Then,

$$\begin{aligned} Pr[A \cap B] &= Pr[A]Pr[B], \text{ so that} \\ \frac{c}{p} &= \frac{a}{p} \times \frac{b}{p}. \text{ Hence,} \\ ab &= cp. \end{aligned}$$

This is not possible since $a, b < p$.

Review: Collisions & Collecting

Collisions:

$$Pr[\text{no collision}] \approx e^{-m^2/2n}$$

Collecting:

$$Pr[\text{miss Wilson}] \approx e^{-m/n}$$

$$Pr[\text{miss at least one}] \leq ne^{-m/n}$$

Review: Math Tricks

Approximations:

$$\ln(1 - \varepsilon) \approx -\varepsilon$$
$$\exp\{-\varepsilon\} \approx 1 - \varepsilon$$

Sums:

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m}$$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2};$$

Math Tricks, continued

Symmetry: E.g., if we pick balls from a bag, with no replacement,

$$Pr[\text{ball 5 is red}] = Pr[\text{ball 1 is red}]$$

Order of balls = permutation.

All permutations have same probability.
Union Bound:

$$Pr[A \cup B \cup C] \leq Pr[A] + Pr[B] + Pr[C]$$

Inclusion/Exclusion:

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

Total Probability:

$$Pr[B] = Pr[A_1]Pr[B|A_1] + \cdots + Pr[A_n]Pr[B|A_n]$$

An L^2-bounded martingale converges almost surely. Just kidding!

A mini-quizz

True or False:

- $Pr[A \cup B] = Pr[A] + Pr[B]$. False True iff disjoint.
- $Pr[A \cap B] = Pr[A]Pr[B]$. False True iff independent.
- $A \cap B = \emptyset \Rightarrow A, B$ independent. False
- For all A, B, one has $Pr[A|B] \geq Pr[A]$. False
- $Pr[A \cap B \cap C] = Pr[A]Pr[B|A]Pr[C|B]$. False

A mini-quizz; part 2

- $\Omega = \{1,2,3,4\}$, uniform. Find events A, B, C that are pairwise independent, not mutually.

  $A = \{1,2\}, B = \{1,3\}, C = \{1,4\}$.
- A, B, C pairwise independent. Is it true that $(A \cap B)$ and C are independent?

  No. In example above, $Pr[A \cap B \cap C] \neq Pr[A \cap B]Pr[C]$.
- Assume $Pr[C|A] > Pr[C|B]$.

  Is it true that $Pr[A|C] > Pr[B|C]$?

  No.
- Deal two cards from a 52-card deck. What is the probability that the value of the first card is strictly larger than that of the second?

  $Pr[\text{same}] = \frac{3}{51}$. $Pr[\text{different}] = \frac{48}{51}$.
  $Pr[\text{first} > \text{second}] = \frac{24}{51}$.