CS70: Jean Walrand: Lecture 26.

Expectation; Geometric & Poisson

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1. Random Variables: Brief Review
2. Expectation
3. Linearity of Expectation
4. Geometric Distribution
5. Poisson Distribution

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Thus, if $V = g(X, Y, Z)$, then $V(\omega) := g(X(\omega), Y(\omega), Z(\omega))$.

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The random variable X is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

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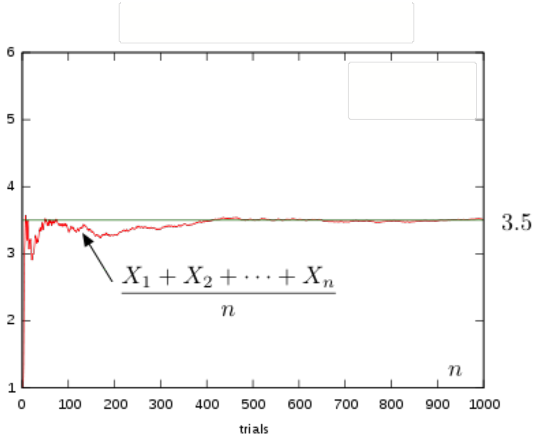

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Note that linearity holds even though the X_m are not independent (whatever that means).

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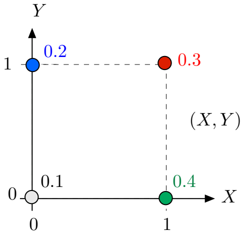

$$(X,Y) = \begin{cases} (0,0), & \text{w.p. } 0.1 \\ (1,0), & \text{w.p. } 0.4 \\ (0,1), & \text{w.p. } 0.2 \\ (1,1), & \text{w.p. } 0.3 \end{cases}$$

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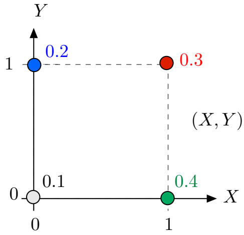

$$(X,Y) = \begin{cases} (0,0), & \text{w.p. } 0.1 \\ (1,0), & \text{w.p. } 0.4 \\ (0,1), & \text{w.p. } 0.2 \\ (1,1), & \text{w.p. } 0.3 \end{cases}$$

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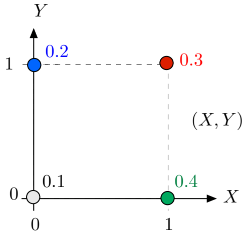

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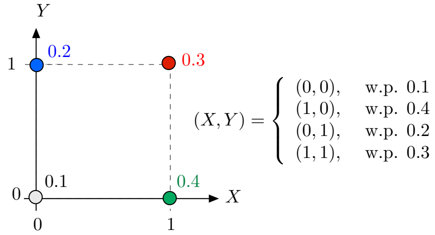

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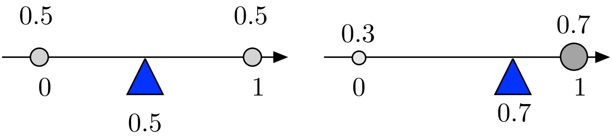
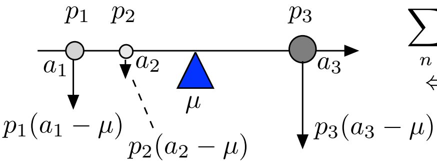
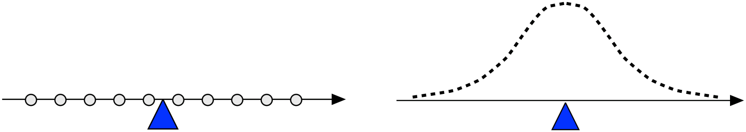




$$\sum_n p_n(a_n - \mu) = 0$$

$$\Leftrightarrow \mu = \sum_n a_n p_n = E[X]$$



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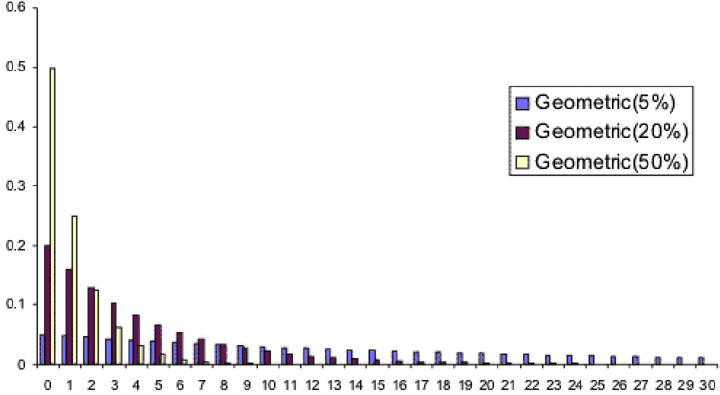

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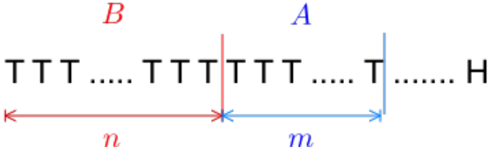

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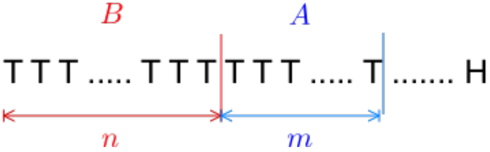

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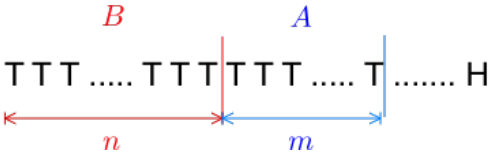

$$Pr[X > n+m | X > n] = Pr[A|B] = Pr[A] = Pr[X > m].$$

The coin is memoryless, therefore, so is X.

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Theorem: For a r.v. X that takes the values $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

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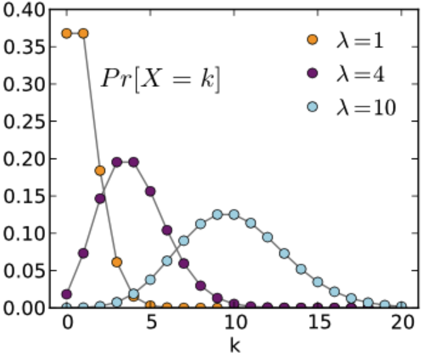

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For (1) we used $m \ll n$; for (2) we used $(1 - a/n)^n \approx e^{-a}$.

Poisson Distribution: Definition and Mean

Definition Poisson Distribution with parameter $\lambda > 0$

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$\square$

Simeon Poisson

The Poisson distribution is named after:

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Siméon Poisson

Siméon Denis Poisson (1781–1840)

Equal Time: B. Geometric

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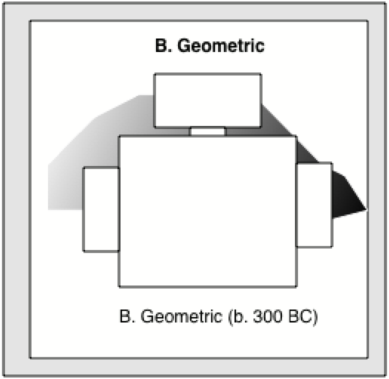

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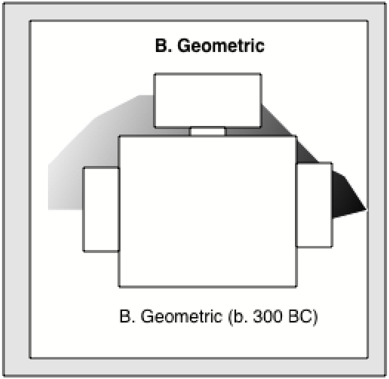

I could not find a picture of D. Binomial, sorry.

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Expectation; Geometric & Poisson

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