CS70: Lecture 27.

Coupons; Independent Random Variables

1. Time to Collect Coupons
2. Review: Independence of Events
3. Independent RVs
4. Mutually independent RVs

Coupon Collectors Problem.

Experiment: Get coupons at random from n until collect all n coupons.
Outcomes: $\{123145..., 56765...\}$
Random Variable: X - length of outcome.

Before: $Pr[X \geq n \ln 2n] \leq \frac{1}{2}$.

Today: $E[X]$?

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second coupon after getting first.

Pr[“get second coupon”|“got ~~milk~~ first coupon”] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Pr[“getting ith coupon|“got $i-1$rst coupons”] $= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n$.

$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(\ln n + \gamma) \end{aligned}$$

Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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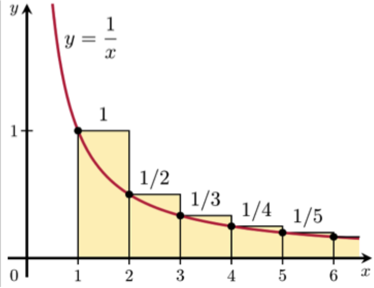
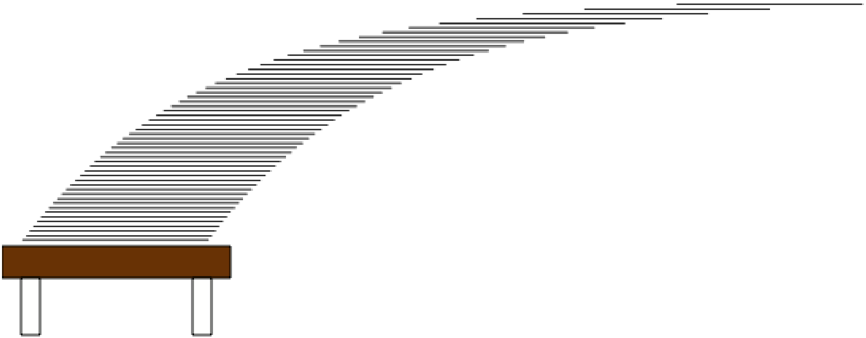


A good approximation is

$$H(n) \approx \ln(n) + \gamma \text{ where } \gamma \approx 0.58 \text{ (Euler-Mascheroni constant).}$$

Harmonic sum: Paradox

Consider this stack of cards (no glue!):



If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Paradox

par·a·dox

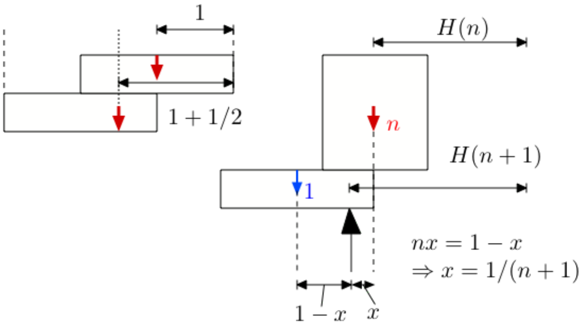
/ˈperəˌdäks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
  synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

Stacking



The cards have width 2. Induction shows that the center of gravity after *n* cards is $H(n)$ away from the right-most edge.

Review: Independence of Events

- Events A, B are independent if $Pr[A \cap B] = Pr[A]Pr[B]$.
- Events A, B, C are mutually independent if

  A, B are independent, A, C are independent, B, C are independent

  and $Pr[A \cap B \cap C] = Pr[A]Pr[B]Pr[C]$.
- Events $\{A_n, n \geq 0\}$ are mutually independent if ....
- Example: $X, Y \in \{0, 1\}$ two fair coin flips $\Rightarrow X, Y, X \oplus Y$ are pairwise independent but not mutually independent.
- Example: $X, Y, Z \in \{0, 1\}$ three fair coin flips are mutually independent.

Independent Random Variables.

Definition: Independence

The random variables X and Y are **independent** if and only if

$$Pr[Y = b | X = a] = Pr[Y = b], \text{ for all } a \text{ and } b.$$

Fact:

X, Y are independent if and only if

$$Pr[X = a, Y = b] = Pr[X = a] Pr[Y = b], \text{ for all } a \text{ and } b.$$

Obvious.

Independence: Examples

Example 1
Roll two die. X, Y = number of pips on the two dice. X, Y are independent.

Indeed: $Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}$.

Example 2
Roll two die. X = total number of pips, Y = number of pips on die 1 minus number on die 2. X and Y are not independent.

Indeed: $Pr[X = 12, Y = 1] = 0 \neq Pr[X = 12]Pr[Y = 1] > 0$.

Example 3
Flip a fair coin five times, X = number of Hs in first three flips, Y = number of Hs in last two flips. X and Y are independent.

Indeed:

$$Pr[X = a, Y = b] = \binom{3}{a}\binom{2}{b}2^{-5} = \binom{3}{a}2^{-3} \times \binom{2}{b}2^{-2} = Pr[X = a]Pr[Y = b].$$

A useful observation about independence

Theorem

X and Y are independent if and only if

$$Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B] \text{ for all } A, B \subset \Re.$$

Proof:

If ($\Leftarrow$): Choose $A = \{a\}$ and $B = \{b\}$.

This shows that $Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b]$.

Only if ($\Rightarrow$):

$$\begin{aligned}
&Pr[X \in A, Y \in B] \\
&= \sum_{a \in A} \sum_{b \in B} Pr[X = a, Y = b] = \sum_{a \in A} \sum_{b \in B} Pr[X = a]Pr[Y = b] \\
&= \sum_{a \in A} [\sum_{b \in B} Pr[X = a]Pr[Y = b]] = \sum_{a \in A} Pr[X = a][\sum_{b \in B} Pr[Y = b]] \\
&= \sum_{a \in A} Pr[X = a]Pr[Y \in B] = Pr[X \in A]Pr[Y \in B].
\end{aligned}$$

$\square$

Functions of Independent random Variables

Theorem Functions of independent RVs are independent
Let X, Y be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$

Proof:
Recall the definition of inverse image:

$$h(z) \in C \Leftrightarrow z \in h^{-1}(C) := \{z \mid h(z) \in C\}. \tag{1}$$

Now,

$$\begin{aligned}
&Pr[f(X) \in A, g(Y) \in B] \\
&\quad = Pr[X \in f^{-1}(A), Y \in g^{-1}(B)], \text{ by } (\textbf{??}) \\
&\quad = Pr[X \in f^{-1}(A)] Pr[Y \in g^{-1}(B)], \text{ since } X, Y \text{ ind.} \\
&\quad = Pr[f(X) \in A] Pr[g(Y) \in B], \text{ by } (\textbf{??}).
\end{aligned}$$

$\square$

Mean of product of independent RV

Theorem
Let X, Y be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

Proof:
Recall that $E[g(X,Y)] = \sum_{x,y} g(x,y)Pr[X = x, Y = y]$. Hence,

$$\begin{aligned}
E[XY] &= \sum_{x,y} xyPr[X = x, Y = y] = \sum_{x,y} xyPr[X = x]Pr[Y = y], \text{ by ind.} \\
&= \sum_{x}[\sum_{y} xyPr[X = x]Pr[Y = y]] = \sum_{x}[xPr[X = x](\sum_{y} yPr[Y = y])] \\
&= \sum_{x}[xPr[X = x]E[Y]] = E[X]E[Y].
\end{aligned}$$

$\square$

Examples

(1) Assume that X, Y, Z are (pairwise) independent, with $E[X] = E[Y] = E[Z] = 0$ and $E[X^2] = E[Y^2] = E[Z^2] = 1$.

Then

$$
\begin{aligned}
&E[(X+2Y+3Z)^2] = E[X^2+4Y^2+9Z^2+4XY+12YZ+6XZ] \\
&\quad = 1+4+9+4\times 0+12\times 0+6\times 0 \\
&\quad = 14.
\end{aligned}
$$

(2) Let X, Y be independent and $U[1, 2, \ldots n]$. Then

$$
\begin{aligned}
E[(X-Y)^2] &= E[X^2+Y^2-2XY] = 2E[X^2]-2E[X]^2 \\
&= \frac{1+3n+2n^2}{3} - \frac{(n+1)^2}{2}.
\end{aligned}
$$

Mutually Independent Random Variables

Definition

X, Y, Z are mutually independent if

$$Pr[X = x, Y = y, Z = z] = Pr[X = x]Pr[Y = y]Pr[Z = z], \text{ for all } x, y, z.$$

Theorem
The events $A, B, C, \ldots$ are pairwise (resp. mutually) independent iff the random variables $1_A, 1_B, 1_C, \ldots$ are pairwise (resp. mutually) independent.
Proof:

$$Pr[1_A = 1, 1_B = 1, 1_C = 1] = Pr[A \cap B \cap C], \ldots$$

$\square$

Functions of pairwise independent RVs

If X, Y, Z are pairwise independent, but not mutually independent, it may be that

$$f(X) \text{ and } g(Y, Z) \text{ are not independent.}$$

Example 1: Flip two fair coins,
$X = 1\{\text{coin 1 is } H\}, Y = 1\{\text{coin 2 is } H\}, Z = X \oplus Y$. Then, X, Y, Z are pairwise independent. Let $g(Y, Z) = Y \oplus Z$. Then $g(Y, Z) = X$ is not independent of X.

Example 2: Let A, B, C be pairwise but not mutually independent in a way that A and $B \cap C$ are not independent. Let
$X = 1_A, Y = 1_B, Z = 1_C$. Choose $f(X) = X, g(Y, Z) = YZ$.

A Little Lemma

Let $X_1, X_2, \ldots, X_{11}$ be mutually independent random variables. Define $Y_1 = (X_1, \ldots, X_4)$, $Y_2 = (X_5, \ldots, X_8)$, $Y_3 = (X_9, \ldots, X_{11})$. Then

$$Pr[Y_1 \in B_1, Y_2 \in B_2, Y_3 \in B_3] = Pr[Y_1 \in B_1] Pr[Y_2 \in B_2] Pr[Y_3 \in B_3].$$

Proof:

$$\begin{aligned}
&Pr[Y_1 \in B_1, Y_2 \in B_2, Y_3 \in B_3] \\
&\quad = \sum_{y_1 \in B_1, y_2 \in B_2, y_3 \in B_3} Pr[Y_1 = y_1, Y_2 = y_2, Y_3 = y_3] \\
&\quad = \sum_{y_1 \in B_1, y_2 \in B_2, y_3 \in B_3} Pr[Y_1 = y_1] Pr[Y_2 = y_2] Pr[Y_3 = y_3] \\
&\quad = \{\sum_{y_1 \in B_1} Pr[Y_1 = y_1]\}\{\sum_{y_2 \in B_2} Pr[Y_2 = y_2]\}\{\sum_{y_3 \in B_3} Pr[Y_3 = y_3]\} \\
&\quad = Pr[Y_1 \in B_1] Pr[Y_2 \in B_2] Pr[Y_3 \in B_3]. \quad \square
\end{aligned}$$

Functions of mutually independent RVs

One has the following result:

Theorem

Functions of disjoint collections of mutually independent random variables are mutually independent.

Example:

Let $\{X_n, n \geq 1\}$ be mutually independent. Then,

$Y_1 := X_1 X_2 (X_3 + X_4)^2, Y_2 := \max\{X_5, X_6\} - \min\{X_7, X_8\}, Y_3 := X_9 \cos(X_{10} + X_{11})$

are mutually independent.

Proof:

Let $B_1 := \{(x_1, x_2, x_3, x_4) \mid x_1 x_2 (x_3 + x_4)^2 \in A_1\}$. Similarly for B_2, B_2.

Then

$$
\begin{aligned}
& Pr[Y_1 \in A_1, Y_2 \in A_2, Y_3 \in A_3] \\
& \quad = Pr[(X_1, \ldots, X_4) \in B_1, (X_5, \ldots, X_8) \in B_2, (X_9, \ldots, X_{11}) \in B_3] \\
& \quad = Pr[(X_1, \ldots, X_4) \in B_1] Pr[(X_5, \ldots, X_8) \in B_2] Pr[(X_9, \ldots, X_{11}) \in B_3] \\
& \qquad \qquad \text{by little lemma} \\
& \quad = Pr[Y_1 \in A_1] Pr[Y_2 \in A_2] Pr[Y_3 \in A_3] \quad \square
\end{aligned}
$$

Operations on Mutually Independent Events

Theorem

Operations on disjoint collections of mutually independent events produce mutually independent events.

For instance, if A, B, C, D, E are mutually independent, then $A\Delta B, C \setminus D, \bar{E}$ are mutually independent.

Proof:

$1_{A\Delta B} = f(1_A, 1_B)$ where
$f(0,0) = 0, f(1,0) = 1, f(0,1) = 1, f(1,1) = 0$

$1_{C\setminus D} = g(1_C, 1_D)$ where
$g(0,0) = 0, g(1,0) = 1, g(0,1) = 0, g(1,1) = 0$

$1_{\bar{E}} = h(1_E)$ where
$h(0) = 1$ and $h(1) = 0$.

Hence, $1_{A\Delta B}, 1_{C\setminus D}, 1_{\bar{E}}$ are functions of mutually independent RVs. Thus, those RVs are mutually independent. Consequently, the events of which they are indicators are mutually independent. $\square$

Product of mutually independent RVs

Theorem
Let $X_1, \ldots, X_n$ be mutually independent RVs. Then,

$$E[X_1 X_2 \cdots X_n] = E[X_1]E[X_2] \cdots E[X_n].$$

Proof:

Assume that the result is true for n. (It is true for $n = 2$.)

Then, with $Y = X_1 \cdots X_n$, one has

$$\begin{aligned} E[X_1 \cdots X_n X_{n+1}] &= E[Y X_{n+1}], \\ &= E[Y]E[X_{n+1}], \\ &\quad \text{because } Y, X_{n+1} \text{ are independent} \\ &= E[X_1] \cdots E[X_n]E[X_{n+1}]. \end{aligned}$$

$\square$

Summary.

$\boxed{\text{Coupons; Independent Random Variables}}$

- Expected time to collect n coupons is $nH(n) \approx n(\ln n + \gamma)$
- X, Y independent $\Leftrightarrow Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B]$
- Then, $f(X), g(Y)$ are independent

  and $E[XY] = E[X]E[Y]$
- Mutual independence ....
- Functions of mutually independent RVs are mutually independent.