CS70: Lecture 27.

Coupons; Independent Random Variables

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1. Time to Collect Coupons
2. Review: Independence of Events
3. Independent RVs
4. Mutually independent RVs

Coupon Collectors Problem.

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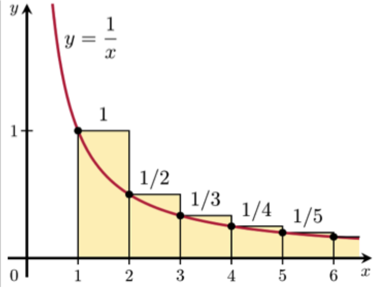

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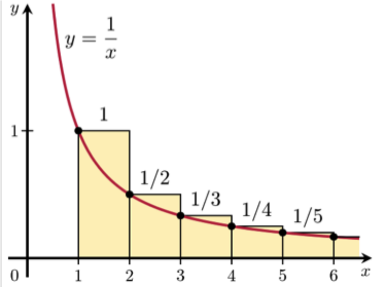

A good approximation is

$$H(n) \approx \ln(n) + \gamma \text{ where } \gamma \approx 0.58 \text{ (Euler-Mascheroni constant).}$$

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Consider this stack of cards (no glue!):

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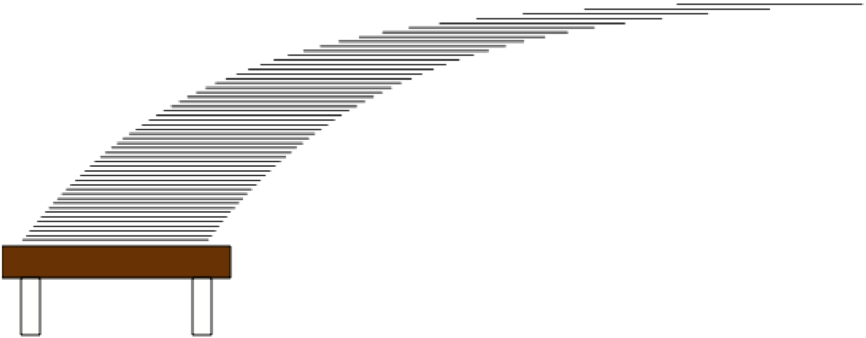

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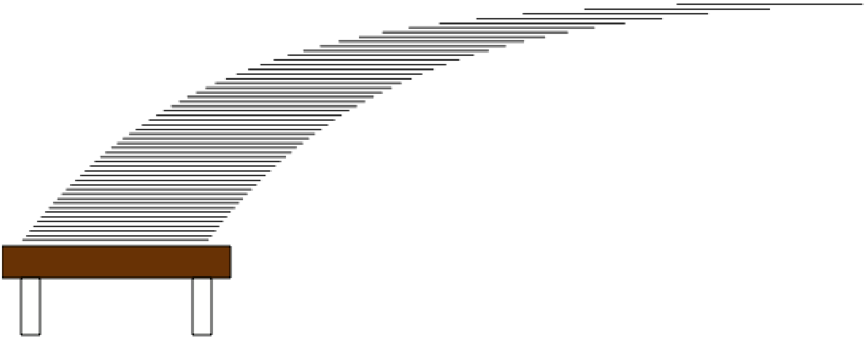

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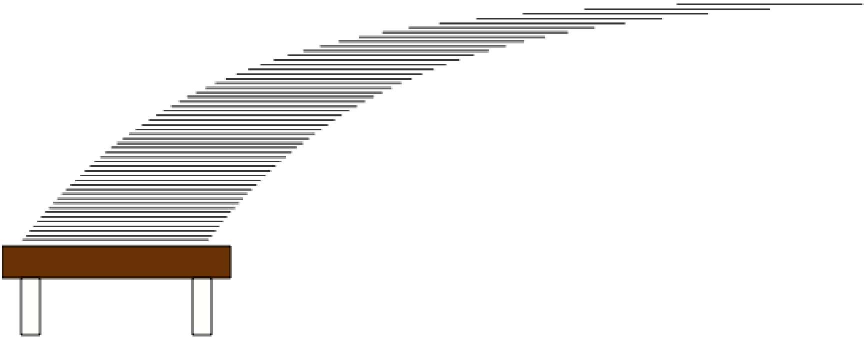


If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Paradox

par·a·dox

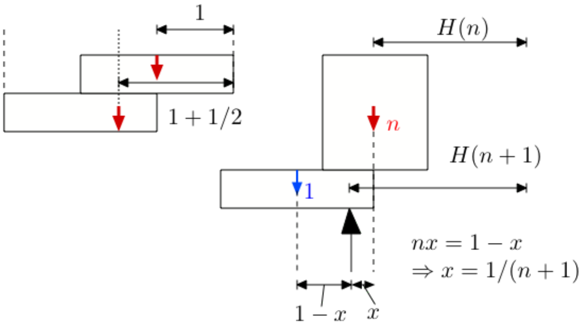
/ˈperəˌdäks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
  synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

Stacking



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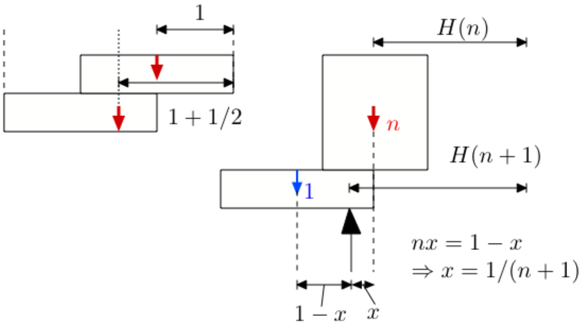

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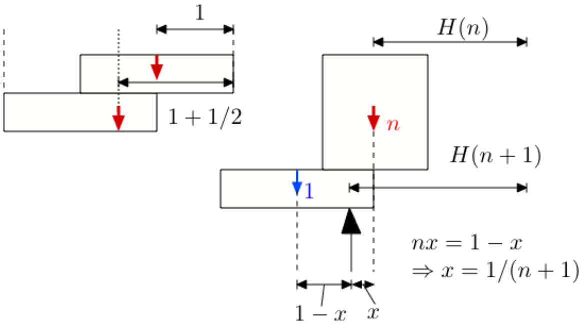

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- Example: $X, Y, Z \in \{0,1\}$ three fair coin flips are mutually independent.

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Let X, Y be independent RV. Then

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Examples

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$$Pr[1_A = 1, 1_B = 1, 1_C = 1] = Pr[A \cap B \cap C], \ldots$$

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Let $X_1, X_2, \ldots, X_{11}$ be mutually independent random variables. Define $Y_1 = (X_1, \ldots, X_4)$, $Y_2 = (X_5, \ldots, X_8)$, $Y_3 = (X_9, \ldots, X_{11})$. Then

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$1_{A\Delta B} = f(1_A, 1_B)$ where
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Coupons; Independent Random Variables

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