CS70: Jean Walrand: Lecture 29.

Confidence Intervals

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1. Confidence?
2. Example
3. Review of Chebyshev
4. Confidence Interval with Chebyshev
5. More examples

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How much confidence do you have in your estimate?

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An estimate without confidence level is useless!

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  - What surgeon do you choose?

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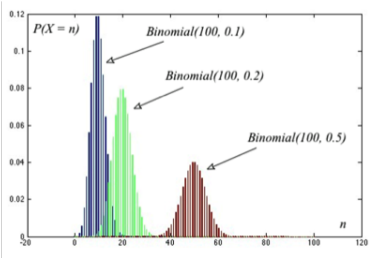

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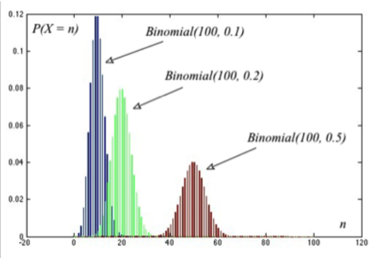

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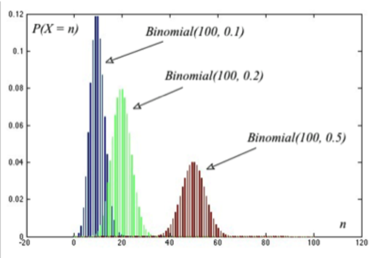

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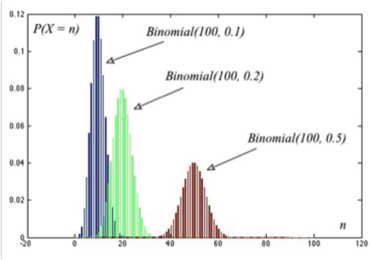

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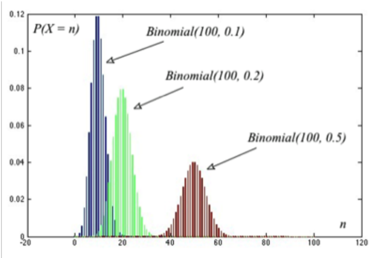

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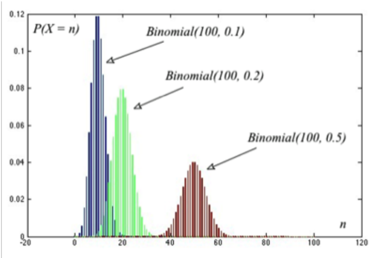

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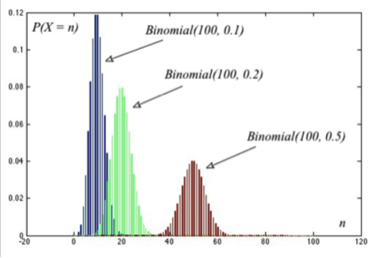

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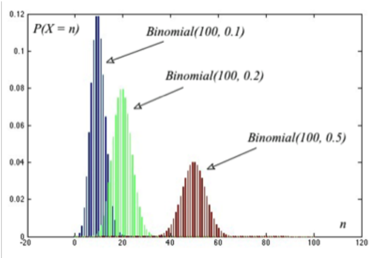

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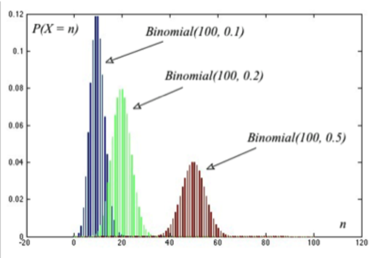

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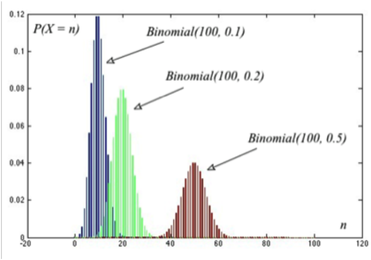

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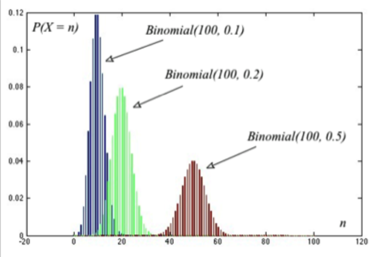

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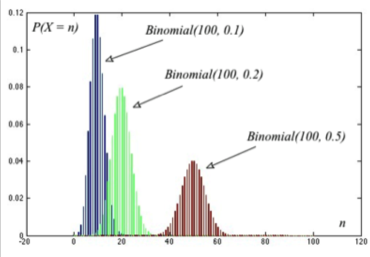

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One approach: Chebyshev.

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$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Confidence Interval with Chebyshev

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Example: If $n = 1500$, then $Pr[p \in [A_n - 0.05, A_n + 0.05]] \geq 95\%$.

In fact, we will see later that $a = \frac{1}{\sqrt{n}}$ works, so that with $n = 1,500$ one has $Pr[p \in [A_n - 0.02, A_n + 0.02]] \geq 95\%$.

Confidence Intervals: Result

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Let X_n be i.i.d. with mean μ and variance σ^2.

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Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1 + \cdots + X_n}{n}$. Then,

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Hence, $[A_n - 4.5\frac{1/2}{\sqrt{n}}, A_n + 4.5\frac{1/2}{\sqrt{n}}]$] is a 95%-CI for p.

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Hence,

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

□

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An illustration:

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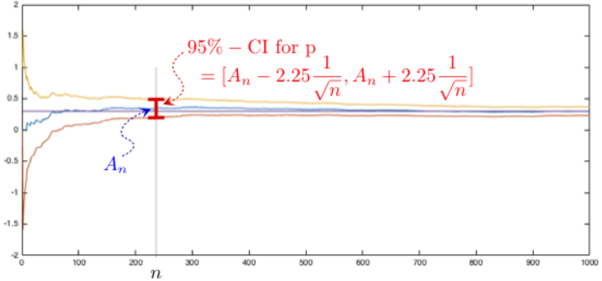

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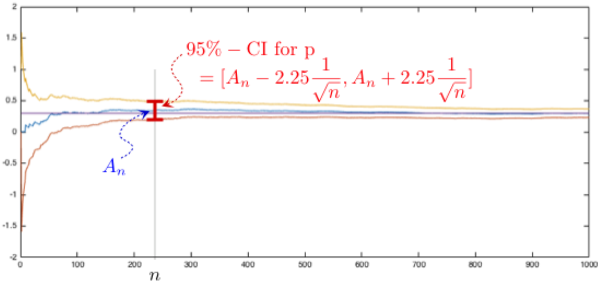

Good practice: You run your simulation, or experiment.

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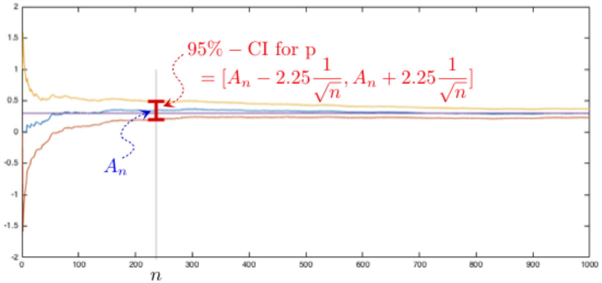

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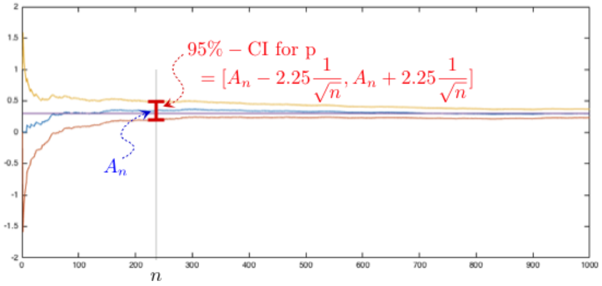

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Confidence interval for p in $B(p)$

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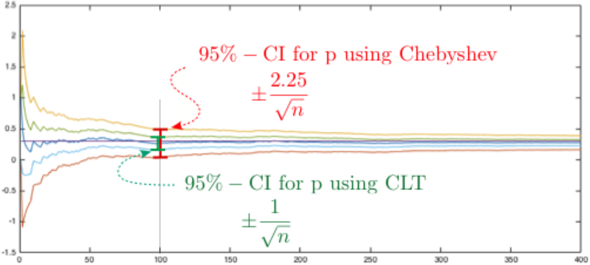
Improved CI:

Confidence interval for p in $B(p)$

Improved CI: Later we will see that we can replace 2.25 by 1.



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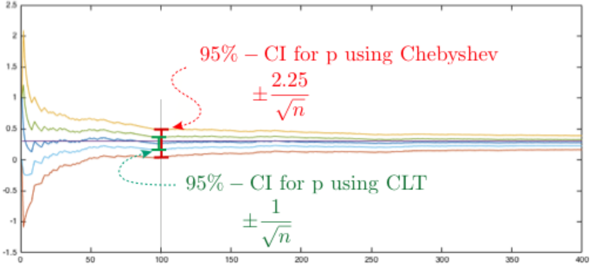

Quite a bit of work to get there:

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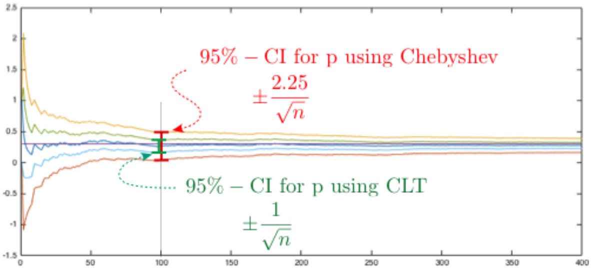

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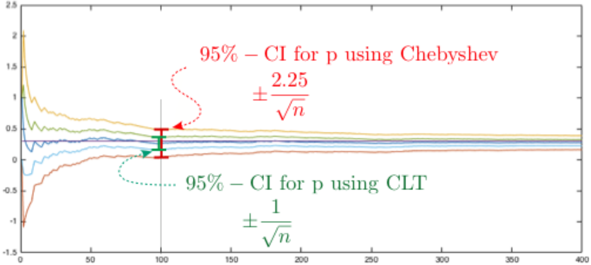

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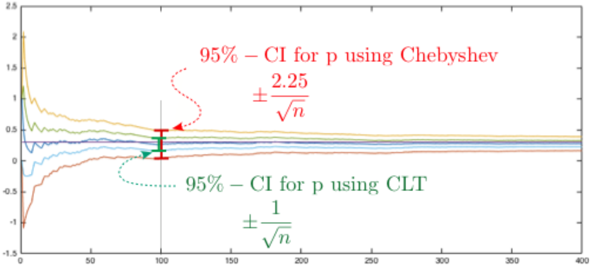

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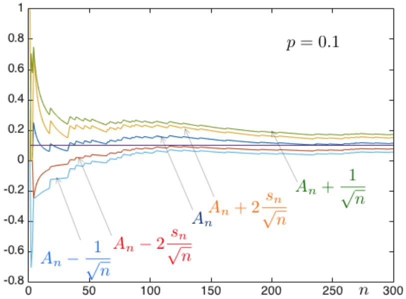

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