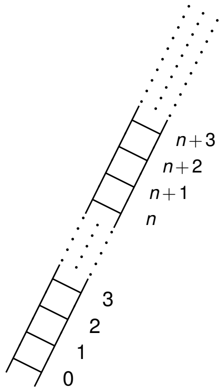
CS70: Lecture 3. Induction!

1. The natural numbers.
2. Seven year old Gauss.
3. ...and Induction.
4. Simple Proof.
5. Two coloring map

(mostly) Next time:

1. Strengthening induction.
2. Tiling Cory Hall courtyard.
3. Horses with one color...

The naturals.



0, 1, 2, 3,
$\ldots, n, n+1, n+2, n+3, \ldots$

A Story about a 7-year old Gauss.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It's 5050! (that is, $\frac{(100)(101)}{2}$)

Gauss and Induction

Child Gauss: $(\forall n \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate for $n = k$. $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!
Induction Step.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}$ **Base Case.**

Statement is true for $n = 0$
plus inductive step $\implies$ true for $n = 1$
plus inductive step $\implies$ true for $n = 2$
...
true for $n = k \implies$ true for $n = k+1$
...

Predicate True for all natural numbers!
Proof by Induction.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

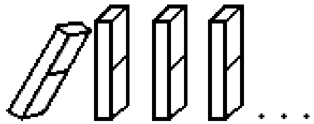
The basic form

- Prove $P(0)$. "Base Case".
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, "Induction Hypothesis"
  - Prove $P(k+1)$. "Induction Step."

$P(n)$ true for all natural numbers n!!!
Get to use $P(k)$ to prove $P(k+1)$! ! !

Notes visualization

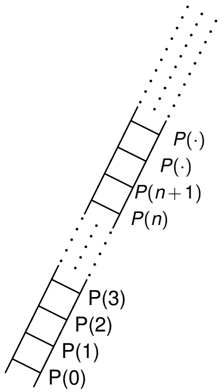
An visualization: an infinite sequence of dominos.



Prove they all fall down;

- $P(0)$ = "First domino falls"
- $(\forall k)\ P(k) \implies P(k+1)$:
  "kth domino falls implies that $k+1$st domino falls"

Climb an infinite ladder?



$$P(0)$$
$$P(k) \implies P(k+1)$$
$$(\forall n \in N) P(n)$$

Your favorite example of "forever"...or the integers...

Simple induction proof.

Theorem: For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does $0 = \frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $1+\cdots+n = \frac{n(n+1)}{2}$

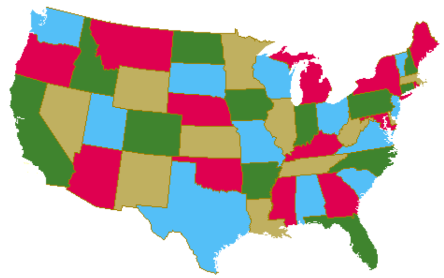
$$\begin{aligned} 1+\cdots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1) \\ &= \frac{n^2+n+2(n+1)}{2} \\ &= \frac{n^2+3n+2}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

Induction Hypothesis.

$P(n+1)$! $(\forall n \in N)\,(P(n) \implies P(n+1))$. $\square$

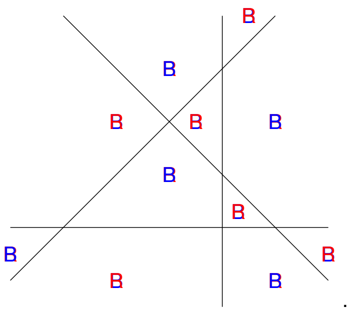
Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.



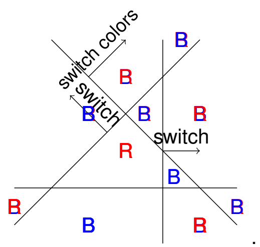
Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.



Fact: Swapping red and blue gives another valid coloring.

Two color theorem: proof illustration.



Base Case.

1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.
   (Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \implies P(k+1)$.

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Variations:

$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

$(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1)))) \implies (\forall n \in N)((n \geq 1) \implies P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!