# CS70: Jean Walrand: Lecture 30.

$\boxed{\text{Linear Regression}}$

1. Preamble
2. Motivation for LR
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4. Linear Regression
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# Linear Regression: Preamble

The best guess about $Y$, if we know only the distribution of $Y$, is $E[Y]$.

More precisely, the value of $a$ that minimizes $E[(Y-a)^2]$ is $a = E[Y]$.

**Proof:**

Let $\hat{Y} := Y - E[Y]$. Then, $E[\hat{Y}] = 0$. So, $E[\hat{Y}c] = 0, \forall c$. Now,

$$\begin{aligned}
E[(Y-a)^2] &= E[(Y - E[Y] + E[Y] - a)^2] \\
&= E[(\hat{Y} + c)^2] \text{ with } c = E[Y] - a \\
&= E[\hat{Y}^2 + 2\hat{Y}c + c^2] = E[\hat{Y}^2] + 2E[\hat{Y}c] + c^2 \\
&= E[\hat{Y}^2] + 0 + c^2 \geq E[\hat{Y}^2].
\end{aligned}$$

Hence, $E[(Y-a)^2] \geq E[(Y - E[Y])^2], \forall a$. $\square$

# Linear Regression: Preamble

Thus, if we want to guess the value of $Y$, we choose $E[Y]$.

Now assume we make some observation $X$ related to $Y$.

How do we use that observation to improve our guess about $Y$?

The idea is to use a function $g(X)$ of the observation to estimate $Y$.

The simplest function $g(X)$ is a constant that does not depend of $X$.

The next simplest function is linear: $g(X) = a + bX$.

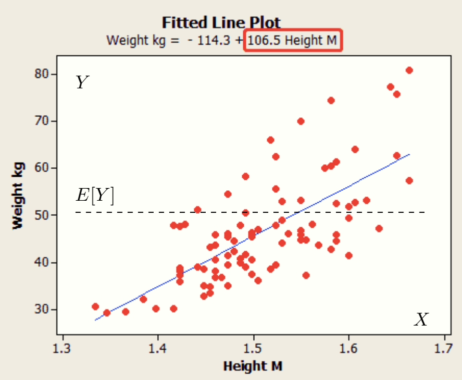
What is the best linear function? That is our next topic.

A bit later, we will consider a general function $g(X)$.

# Linear Regression: Motivation

Example 1: 100 people.

Let $(X_n, Y_n)$ = (height, weight) of person $n$, for $n = 1, \ldots, 100$:



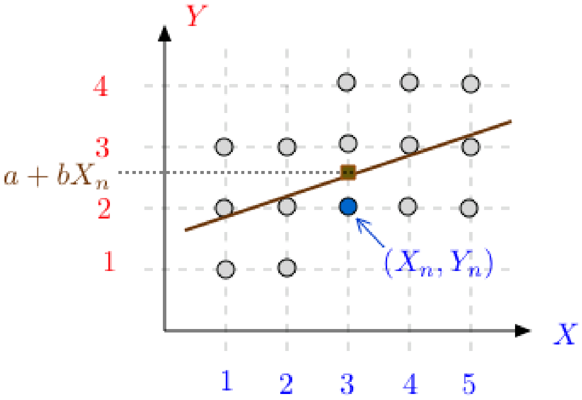

The blue line is $Y = -114.3 + 106.5X$. ($X$ in meters, $Y$ in kg.)

Best linear fit: Linear Regression.

# Motivation

Example 2: 15 people.

We look at two attributes: $(X_n, Y_n)$ of person $n$, for $n = 1, \ldots, 15$:




The line $Y = a + bX$ is the linear regression.

# Covariance

**Definition** The covariance of $X$ and $Y$ is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

**Fact**

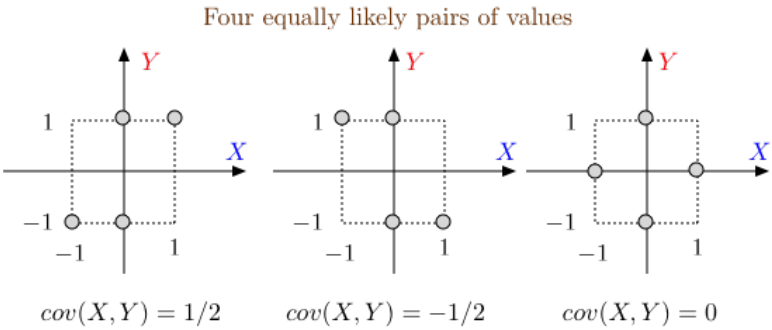
$$cov(X, Y) = E[XY] - E[X]E[Y].$$

**Proof:**

$$\begin{aligned} E[(X - E[X])(Y - E[Y])] &= E[XY - E[X]Y - XE[Y] + E[X]E[Y]] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y]. \end{aligned}$$

$\square$

# Examples of Covariance



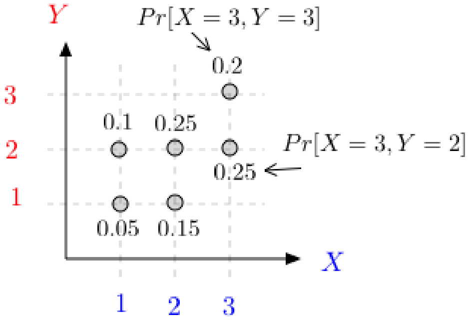

Note that $E[X] = 0$ and $E[Y] = 0$ in these examples. Then $cov(X, Y) = E[XY]$.

When $cov(X, Y) > 0$, the RVs $X$ and $Y$ tend to be large or small together. $X$ and $Y$ are said to be positively correlated.

When $cov(X, Y) < 0$, when $X$ is larger, $Y$ tends to be smaller. $X$ and $Y$ are said to be negatively correlated.

When $cov(X, Y) = 0$, we say that $X$ and $Y$ are uncorrelated.

# Examples of Covariance




$$E[X] = 1 \times 0.15 + 2 \times 0.4 + 3 \times 0.45 = 1.9$$
$$E[X^2] = 1^2 \times 0.15 + 2^2 \times 0.4 + 3^2 \times 0.45 = 5.8$$
$$E[Y] = 1 \times 0.2 + 2 \times 0.6 + 3 \times 0.2 = 2$$
$$E[XY] = 1 \times 0.05 + 1 \times 2 \times 0.1 + \cdots + 3 \times 3 \times 0.2 = 4.85$$
$$cov(X, Y) = E[XY] - E[X]E[Y] = 1.05$$
$$var[X] = E[X^2] - E[X]^2 = 2.19.$$

# Properties of Covariance

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

**Fact**

(a) $var[X] = cov(X,X)$

(b) $X, Y$ independent $\Rightarrow cov(X,Y) = 0$

(c) $cov(a+X, b+Y) = cov(X,Y)$

(d) $cov(aX+bY, cU+dV) = ac.cov(X,U) + ad.cov(X,V) + bc.cov(Y,U) + bd.cov(Y,V).$

**Proof:**

(a)-(b)-(c) are obvious.

(d) In view of (c), one can subtract the means and assume that the RVs are zero-mean. Then,

$$\begin{aligned} cov(aX+bY, cU+dV) &= E[(aX+bY)(cU+dV)] \\ &= ac.E[XU] + ad.E[XV] + bc.E[YU] + bd.E[YV] \\ &= ac.cov(X,U) + ad.cov(X,V) + bc.cov(Y,U) + bd.cov(Y,V). \end{aligned}$$

$\square$

# Linear Regression: Non-Bayesian

**Definition**

Given the samples $\{(X_n, Y_n), n = 1, \ldots, N\}$, the Linear Regression of $Y$ over $X$ is

$$\hat{Y} = a + bX$$

where $(a, b)$ minimize

$$\sum_{n=1}^{N} (Y_n - a - bX_n)^2.$$

Thus, $\hat{Y}_n = a + bX_n$ is our guess about $Y_n$ given $X_n$. The squared error is $(Y_n - \hat{Y}_n)^2$. The LR minimizes the sum of the squared errors.

Why the squares and not the absolute values? Main justification: much easier!

Note: This is a non-Bayesian formulation: there is no prior.

# Linear Least Squares Estimate

**Definition**
Given two RVs $X$ and $Y$ with known distribution $Pr[X = x, Y = y]$, the Linear Least Squares Estimate of $Y$ given $X$ is

$$\hat{Y} = a + bX =: L[Y|X]$$

where $(a, b)$ minimize

$$g(a,b) := E[(Y - a - bX)^2].$$

Thus, $\hat{Y} = a + bX$ is our guess about $Y$ given $X$. The squared error is $(Y - \hat{Y})^2$. The LLSE minimizes the expected value of the squared error.

Why the squares and not the absolute values? Main justification: much easier!

Note: This is a Bayesian formulation: there is a prior.

# LR: Non-Bayesian or Uniform?

Observe that

$$\frac{1}{N}\sum_{n=1}^{N}(Y_n - a - bX_n)^2 = E[(Y - a - bX)^2]$$

where one assumes that

$$(X, Y) = (X_n, Y_n), \text{ w.p. } \frac{1}{N} \text{ for } n = 1, \dots, N.$$

That is, the non-Bayesian LR is equivalent to the Bayesian LLSE that assumes that $(X, Y)$ is uniform on the set of observed samples.

Thus, we can study the two cases LR and LLSE in one shot.

However, the interpretations are different!

# LLSE

**Theorem**
Consider two RVs $X, Y$ with a given distribution $Pr[X = x, Y = y]$. Then,

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$

**Proof 1:**
$Y - \hat{Y} = (Y - E[Y]) - \frac{cov(X,Y)}{var[X]}(X - E[X])$. Hence, $E[Y - \hat{Y}] = 0$.

Also, $E[(Y - \hat{Y})X] = 0$, after a bit of algebra. (See next slide.)

Hence, by combining the two brown equalities, $E[(Y - \hat{Y})(c + dX)] = 0$. Then, $E[(Y - \hat{Y})(\hat{Y} - a - bX)] = 0, \forall a, b$. Indeed: $\hat{Y} = \alpha + \beta X$ for some $\alpha, \beta$, so that $\hat{Y} - a - bX = c + dX$ for some $c, d$. Now,

$$\begin{aligned} E[(Y - a - bX)^2] &= E[(Y - \hat{Y} + \hat{Y} - a - bX)^2] \\ &= E[(Y - \hat{Y})^2] + E[(\hat{Y} - a - bX)^2] + 0 \geq E[(Y - \hat{Y})^2]. \end{aligned}$$

This shows that $E[(Y - \hat{Y})^2] \leq E[(Y - a - bX)^2]$, for all $(a, b)$. Thus $\hat{Y}$ is the LLSE. $\square$

# A Bit of Algebra

$Y - \hat{Y} = (Y - E[Y]) - \frac{cov(X,Y)}{var[X]}(X - E[X])$.

Hence, $E[Y - \hat{Y}] = 0$. We want to show that $E[(Y - \hat{Y})X] = 0$.

Note that

$$E[(Y - \hat{Y})X] = E[(Y - \hat{Y})(X - E[X])],$$

because $E[(Y - \hat{Y})E[X]] = 0$.

Now,

$$\begin{aligned} & E[(Y - \hat{Y})(X - E[X])] \\ & \quad = E[(Y - E[Y])(X - E[X])] - \frac{cov(X,Y)}{var[X]} E[(X - E[X])(X - E[X])] \\ & \quad =^{(*)} cov(X,Y) - \frac{cov(X,Y)}{var[X]} var[X] = 0. \quad \square \end{aligned}$$

$^{(*)}$ Recall that $cov(X,Y) = E[(X - E[X])(Y - E[Y])]$ and $var[X] = E[(X - E[X])^2]$.

# Estimation Error

We saw that the LLSE of $Y$ given $X$ is

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$

How good is this estimator? That is, what is the mean squared estimation error?

We find

$$\begin{aligned}
E[|Y - L[Y|X]|^2] &= E[(Y - E[Y] - (cov(X,Y)/var(X))(X - E[X]))^2] \\
&= E[(Y - E[Y])^2] - 2(cov(X,Y)/var(X))E[(Y - E[Y])(X - E[X])] \\
&\quad + (cov(X,Y)/var(X))^2 E[(X - E[X])^2] \\
&= var(Y) - \frac{cov(X,Y)^2}{var(X)}.
\end{aligned}$$

Without observations, the estimate is $E[Y] = 0$. The error is $var(Y)$. Observing $X$ reduces the error.

# Estimation Error: A Picture

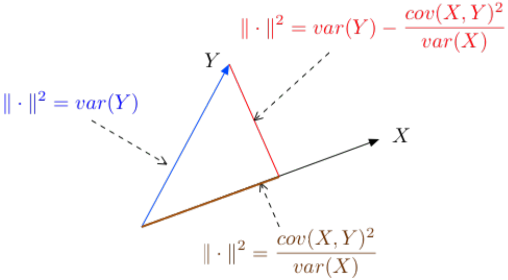
We saw that

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$$

and

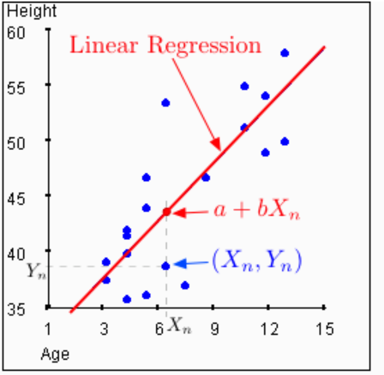
$$E[|Y - L[Y|X]|^2] = var(Y) - \frac{cov(X,Y)^2}{var(X)}.$$

Here is a picture when $E[X] = 0, E[Y] = 0$:



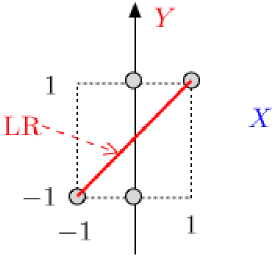
# Linear Regression Examples

Example 1:



# Linear Regression Examples

Example 2:

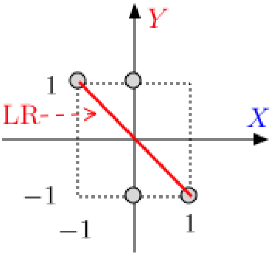



We find:

$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = 1/2;$$
$$var[X] = E[X^2] - E[X]^2 = 1/2; cov(X, Y) = E[XY] - E[X]E[Y] = 1/2;$$
LR: $\hat{Y} = E[Y] + \frac{cov(X, Y)}{var[X]}(X - E[X]) = X.$

# Linear Regression Examples

Example 3:

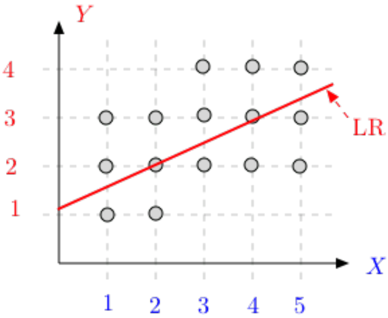



We find:

$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = -1/2;$$
$$var[X] = E[X^2] - E[X]^2 = 1/2; cov(X,Y) = E[XY] - E[X]E[Y] = -1/2;$$
$$\text{LR: } \hat{Y} = E[Y] + \frac{cov(X,Y)}{var[X]}(X - E[X]) = -X.$$

# Linear Regression Examples

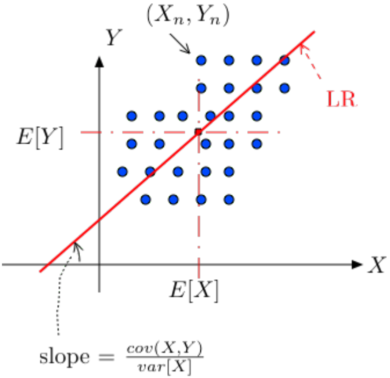
Example 4:




We find:

$$E[X] = 3; E[Y] = 2.5; E[X^2] = (3/15)(1 + 2^2 + 3^2 + 4^2 + 5^2) = 11;$$
$$E[XY] = (1/15)(1 \times 1 + 1 \times 2 + \cdots + 5 \times 4) = 8.4;$$
$$var[X] = 11 - 9 = 2; cov(X, Y) = 8.4 - 3 \times 2.5 = 0.9;$$
$$\text{LR: } \hat{Y} = 2.5 + \frac{0.9}{2}(X - 3) = 1.15 + 0.45X.$$

# LR: Another Figure




Note that

- the LR line goes through $(E[X], E[Y])$
- its slope is $\frac{cov(X,Y)}{var(X)}$.

# Summary

$\boxed{\text{Linear Regression}}$

1. Linear Regression: $L[Y|X] = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$
2. Non-Bayesian: minimize $\sum_n (Y_n - a - bX_n)^2$
3. Bayesian: minimize $E[(Y - a - bX)^2]$