CS70: Jean Walrand: Lecture 30.

Linear Regression

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Linear Regression

1. Preamble
2. Motivation for LR
3. History of LR
4. Linear Regression
5. Derivation
6. More examples

Linear Regression: Preamble

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Hence, $E[(Y-a)^2] \geq E[(Y - E[Y])^2], \forall a.$ $\square$

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A bit later, we will consider a general function $g(X)$.

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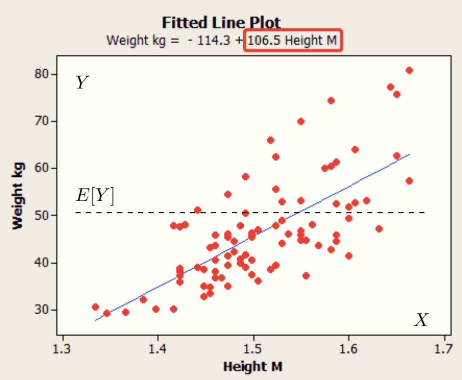

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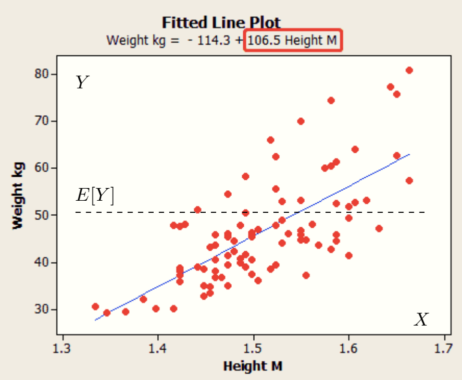

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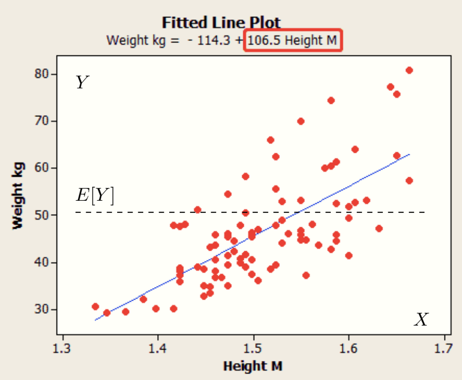

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Best linear fit: Linear Regression.

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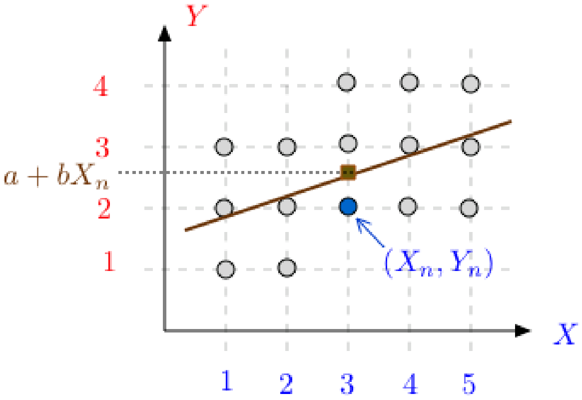

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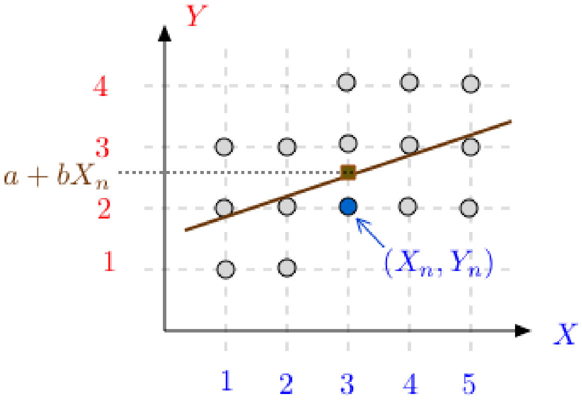

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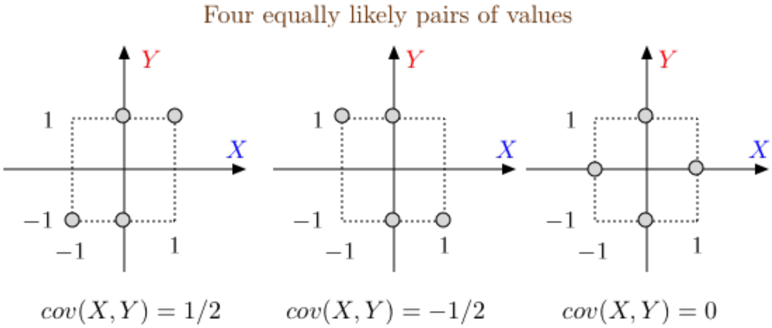
$\square$

Examples of Covariance



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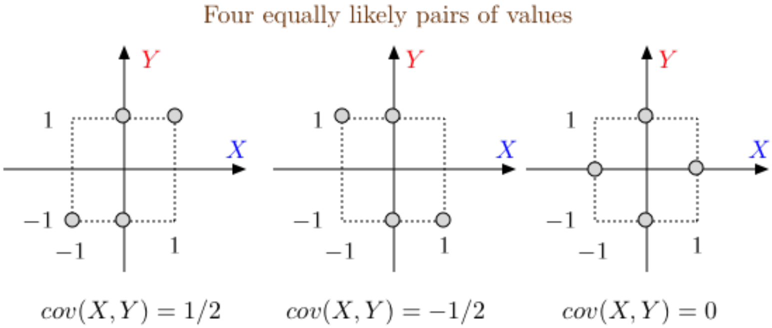

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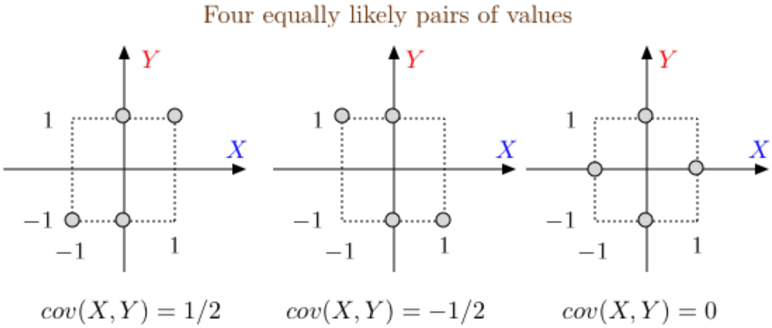

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When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together.

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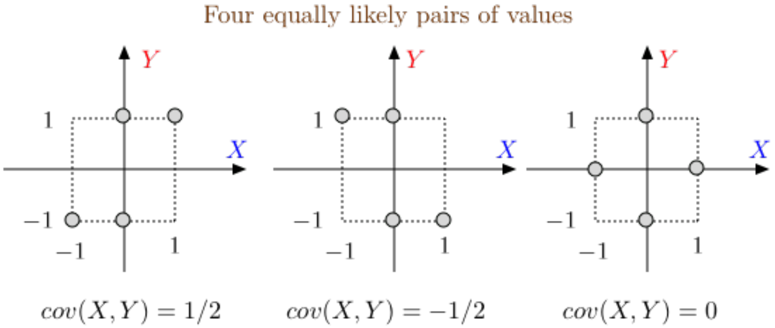

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When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together. X and Y are said to be positively correlated.

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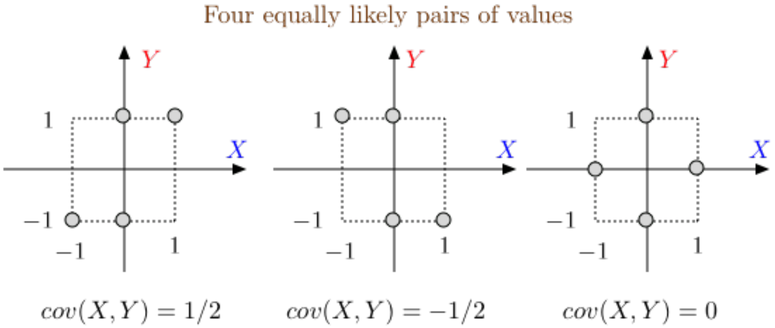

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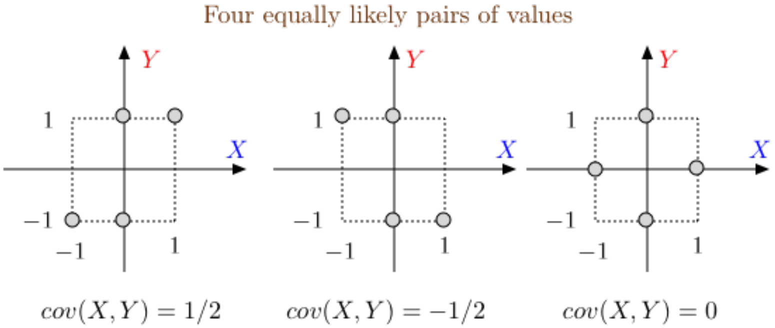

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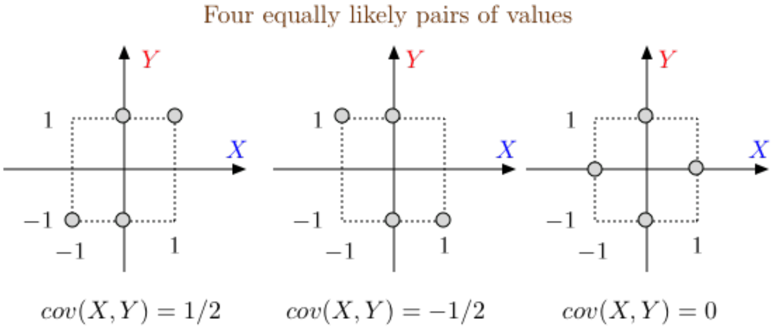

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When $cov(X,Y) = 0$, we say that X and Y are uncorrelated.

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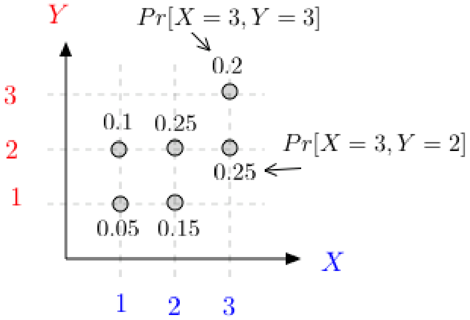

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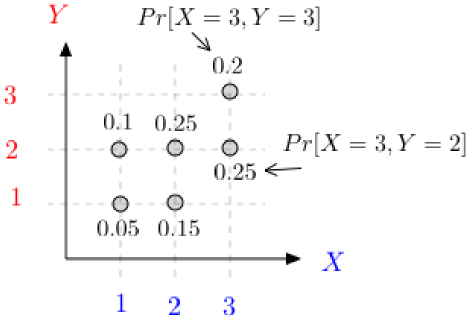

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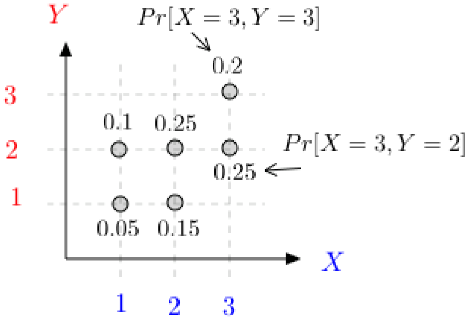

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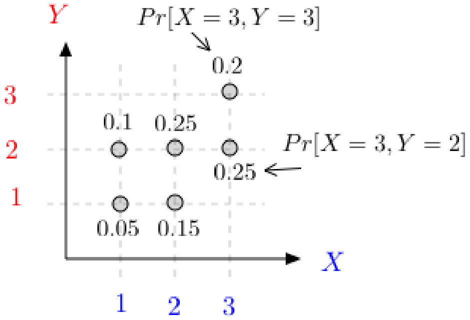

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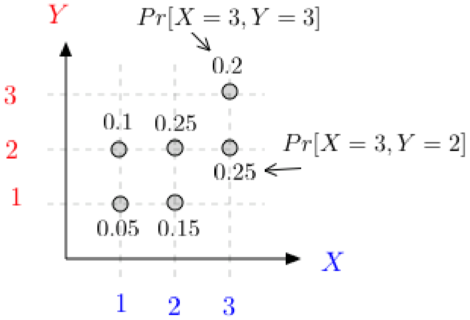

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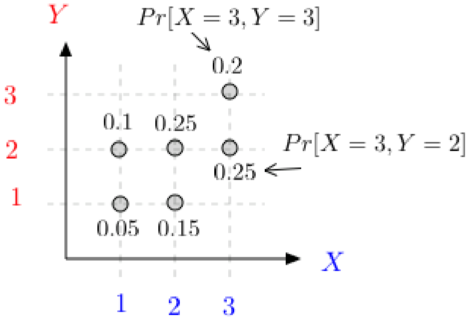

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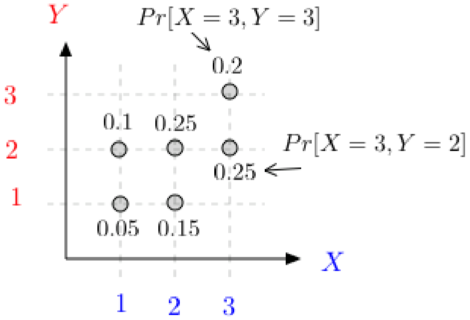

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Properties of Covariance

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(d) $$cov(aX+bY, cU+dV) = ac.cov(X,U) + ad.cov(X,V) + bc.cov(Y,U) + bd.cov(Y,V).$$

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(a)-(b)-(c) are obvious.

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Note: This is a non-Bayesian formulation: there is no prior.

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Note: This is a Bayesian formulation: there is a prior.

LR: Non-Bayesian or Uniform?

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However, the interpretations are different!

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Consider two RVs X, Y with a given distribution $Pr[X = x, Y = y]$. Then,

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Thus $\hat{Y}$ is the LLSE. $\square$

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$$Y - \hat{Y} = (Y - E[Y]) - \frac{cov(X,Y)}{var[X]}(X - E[X]).$$

Hence, $E[Y - \hat{Y}] = 0$. We want to show that $E[(Y - \hat{Y})X] = 0$.

Note that

$$E[(Y - \hat{Y})X] = E[(Y - \hat{Y})(X - E[X])],$$

because $E[(Y - \hat{Y})E[X]] = 0$.

Now,

$$E[(Y - \hat{Y})(X - E[X])]$$
$$= E[(Y - E[Y])(X - E[X])] - \frac{cov(X,Y)}{var[X]} E[(X - E[X])(X - E[X])]$$

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$$\begin{aligned} & E[(Y - \hat{Y})(X - E[X])] \\ & = E[(Y - E[Y])(X - E[X])] - \frac{cov(X,Y)}{var[X]} E[(X - E[X])(X - E[X])] \\ & =^{(*)} cov(X,Y) - \frac{cov(X,Y)}{var[X]} var[X] = 0. \quad \square \end{aligned}$$

$^{(*)}$ Recall that $cov(X,Y) = E[(X - E[X])(Y - E[Y])]$ and $var[X] = E[(X - E[X])^2]$.

Estimation Error

We saw that the LLSE of Y given X is

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Without observations, the estimate is $E[Y] = 0$. The error is $var(Y)$. Observing X reduces the error.

Estimation Error: A Picture

We saw that

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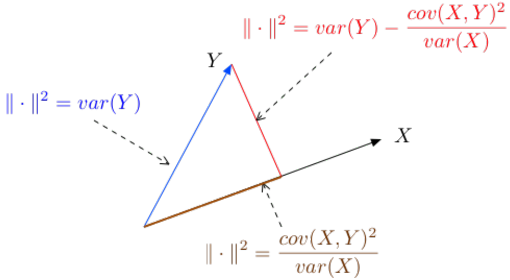

Linear Regression Examples

Example 1:

Linear Regression Examples

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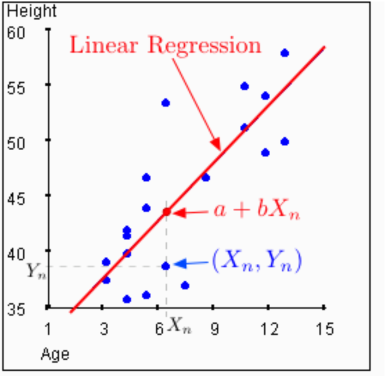

Linear Regression Examples

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Linear Regression Examples

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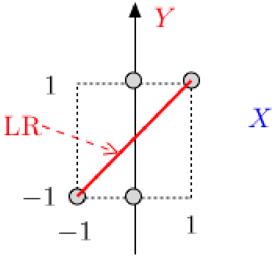

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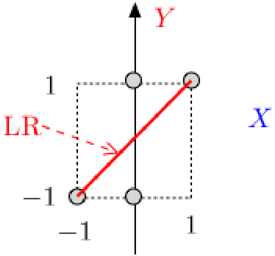

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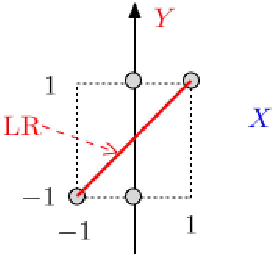

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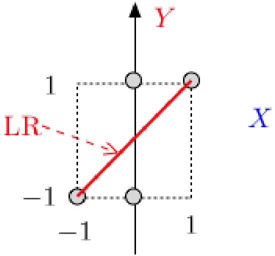

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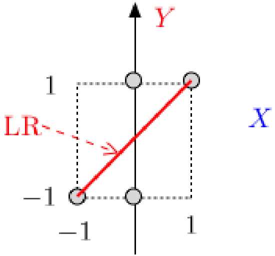

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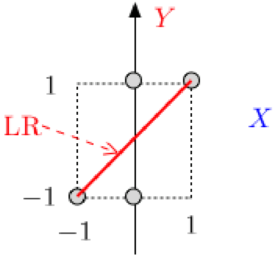

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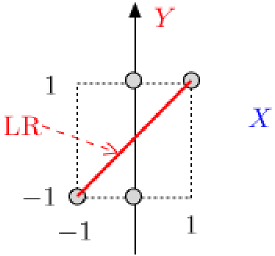

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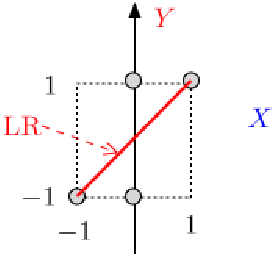

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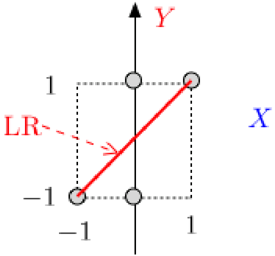

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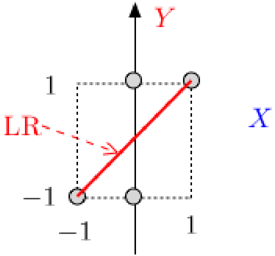

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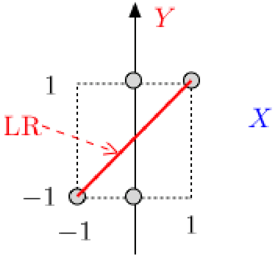

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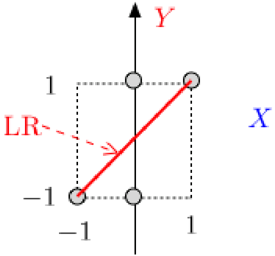

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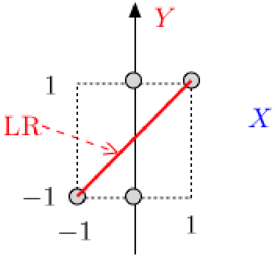

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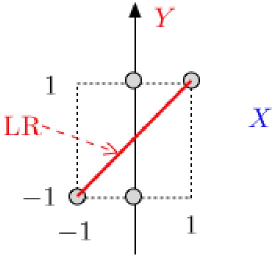

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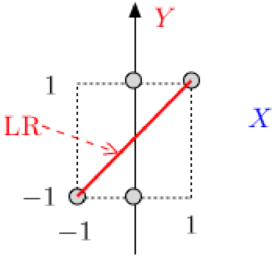

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Linear Regression Examples

Example 3:

Linear Regression Examples

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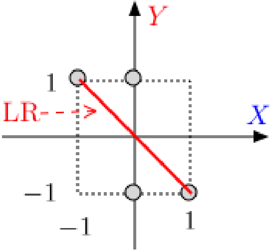

Linear Regression Examples

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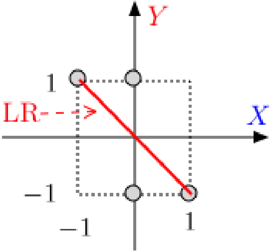

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$E[X] =$

Linear Regression Examples

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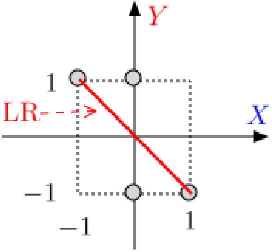

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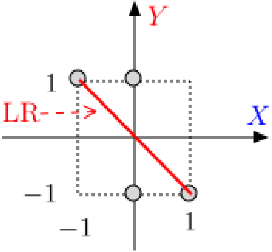

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Linear Regression Examples

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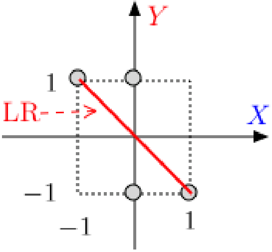

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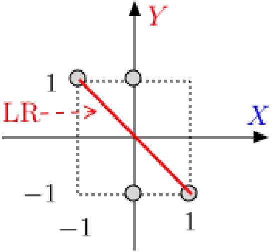

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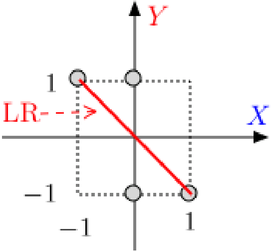

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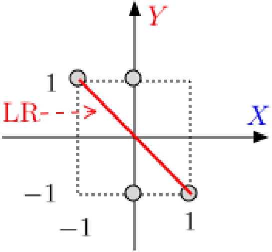

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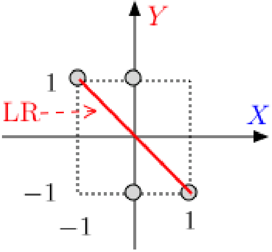

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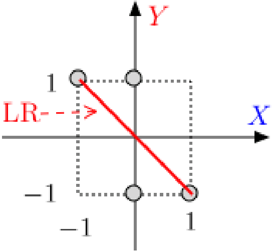

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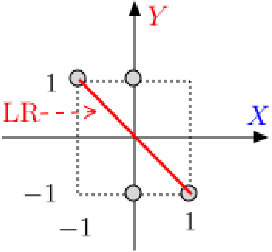

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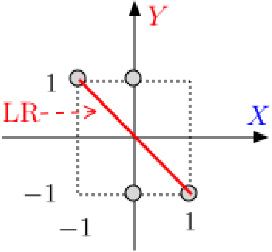

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Linear Regression Examples

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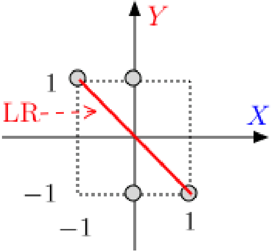

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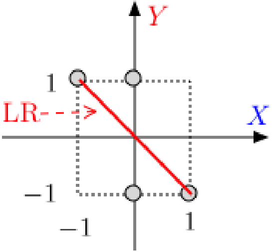

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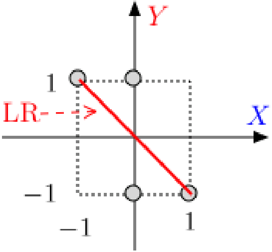

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Linear Regression Examples

Example 4:

Linear Regression Examples

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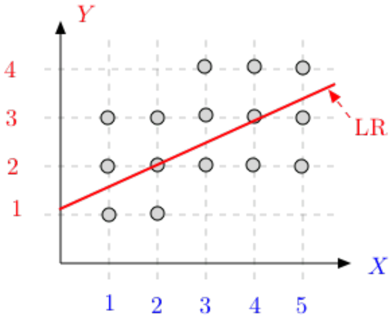

Linear Regression Examples

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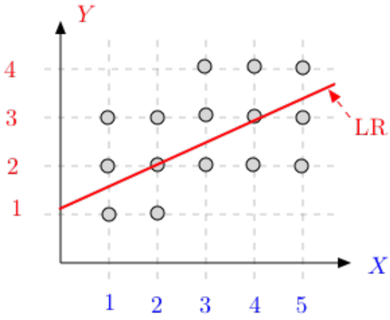

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Linear Regression Examples

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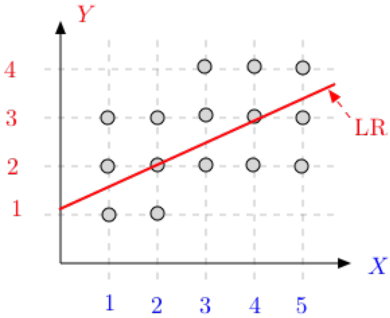

We find:

$E[X] = 3;$

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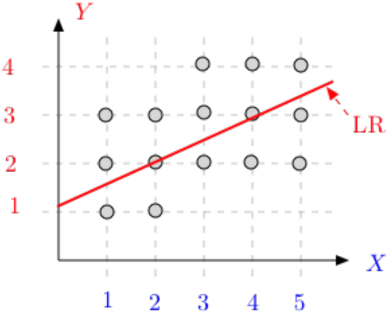

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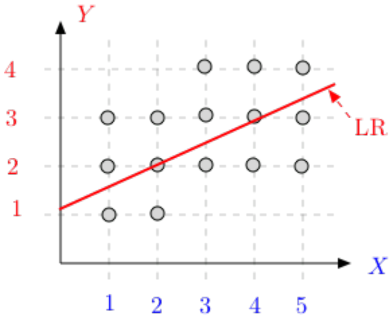

We find:

$$E[X] = 3; E[Y] = 2.5;$$

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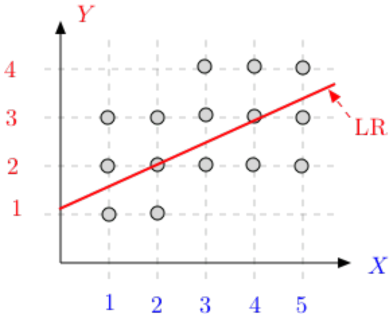

We find:

$$E[X] = 3; E[Y] = 2.5; E[X^2] = (3/15)(1 + 2^2 + 3^2 + 4^2 + 5^2) = 11;$$

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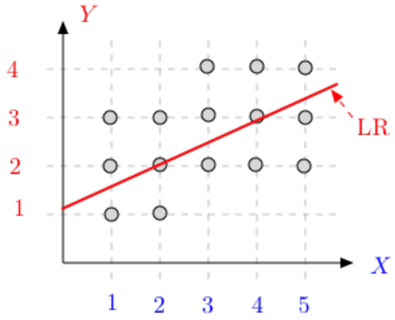

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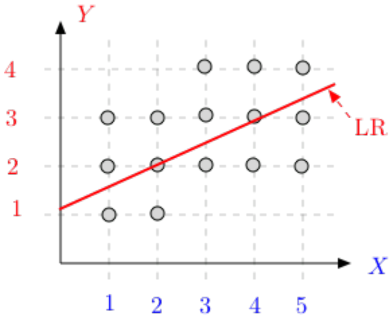

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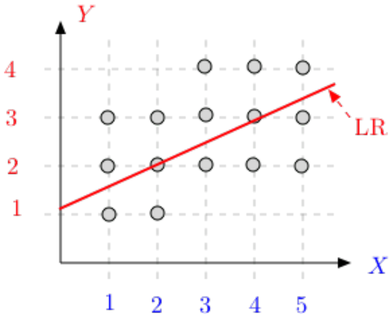

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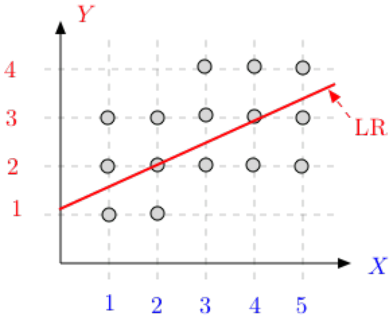

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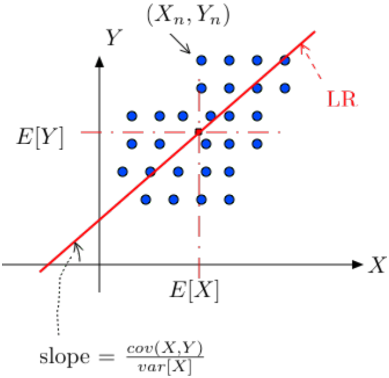
LR: $\hat{Y} = 2.5 + \frac{0.9}{2}(X - 3) = 1.15 + 0.45X.$

LR: Another Figure



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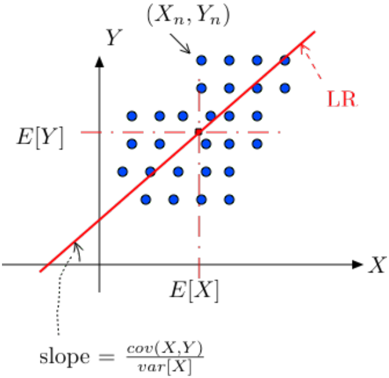

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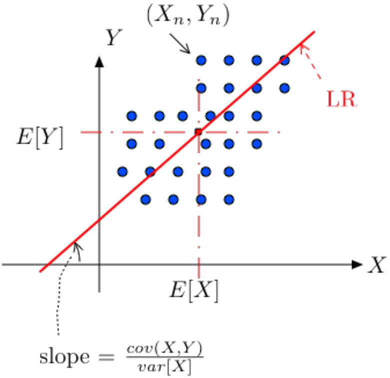

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- the LR line goes through $(E[X], E[Y])$
- its slope is $\frac{cov(X,Y)}{var(X)}$.

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