CS70: Jean Walrand: Lecture 32.

Markov Chains 1

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1. Examples
2. Definition
3. First Passage Time

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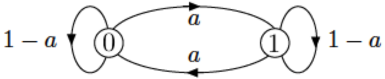

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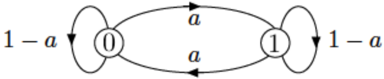

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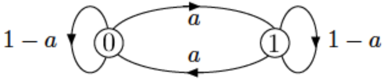

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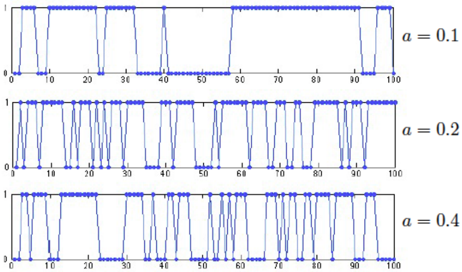

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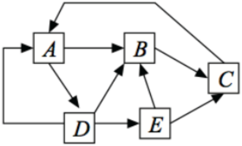
At each step, the MC follows one of the outgoing arrows of the current state, with equal probabilities.



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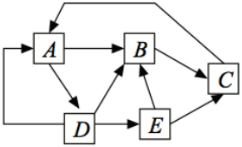

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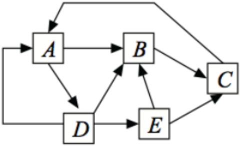

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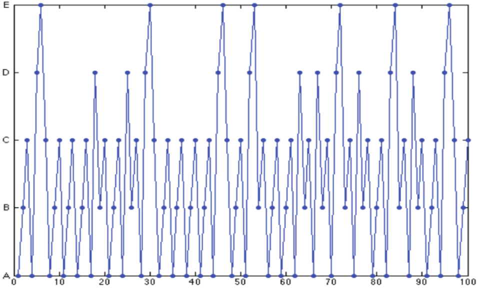

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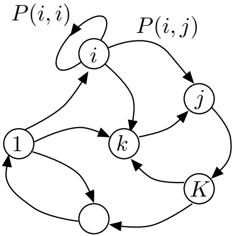

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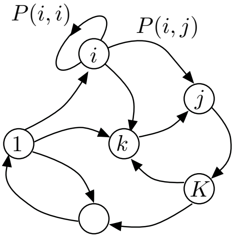

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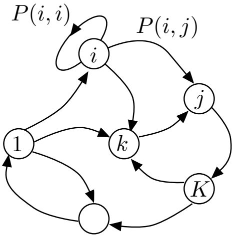

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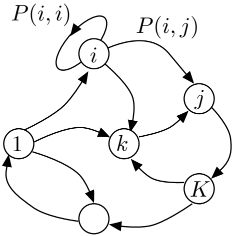

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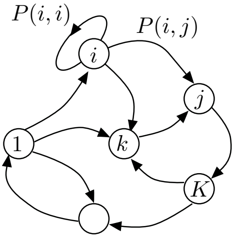

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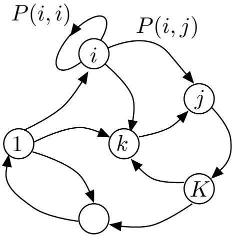

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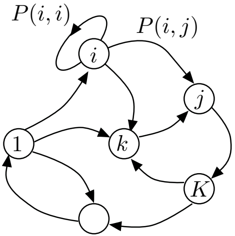

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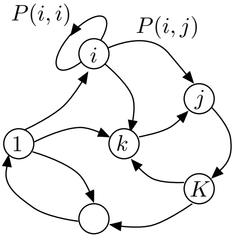

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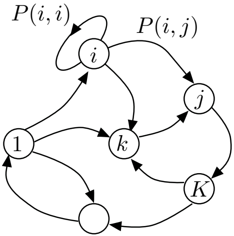

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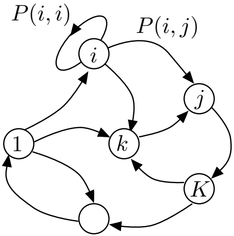

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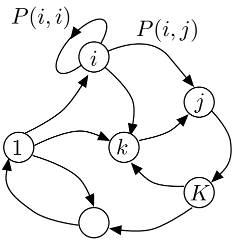

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  $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}$.

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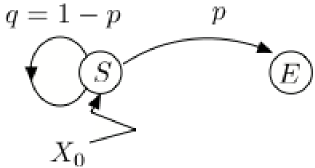

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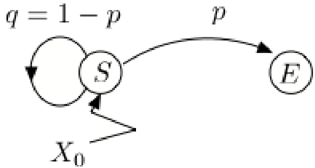

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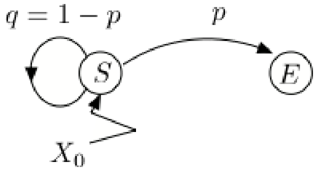

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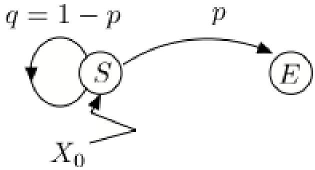

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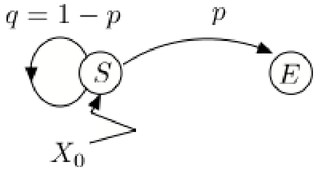

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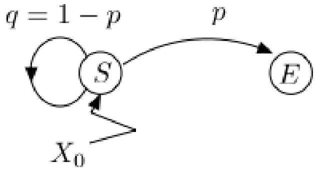

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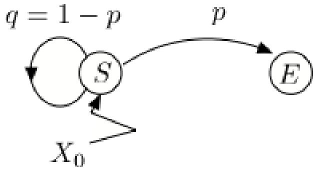

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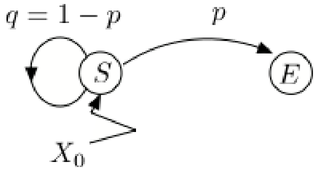

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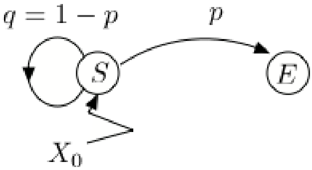

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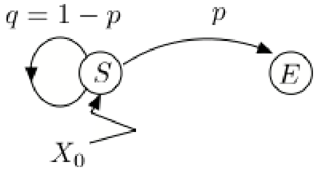

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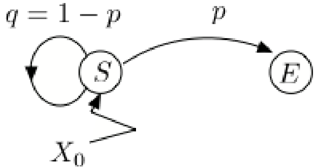

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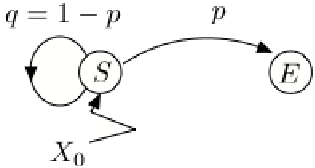

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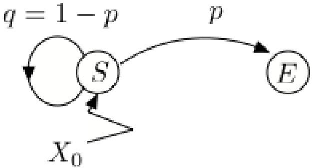

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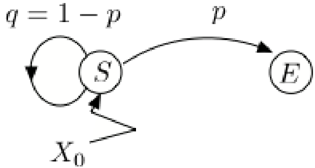

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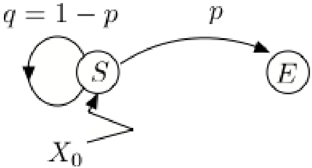

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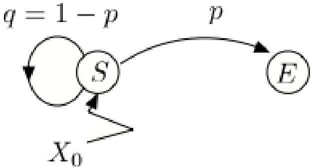

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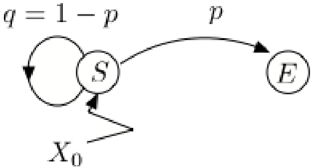

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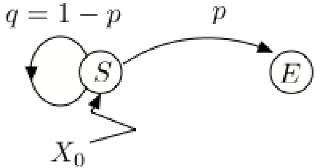

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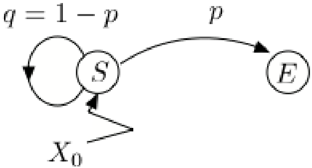

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Now, Z and N' are independent. Also, $E[N'] = E[N] = \beta(S)$. Hence, taking expectation,

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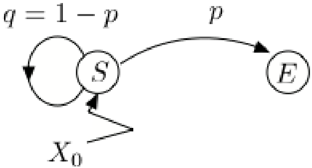
Let's flip a coin with $Pr[H] = p$ until we get H. How many flips, on average?



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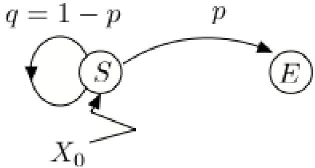

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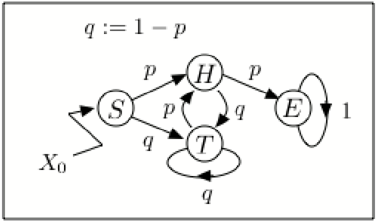

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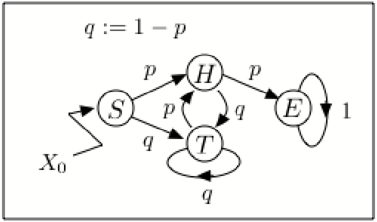

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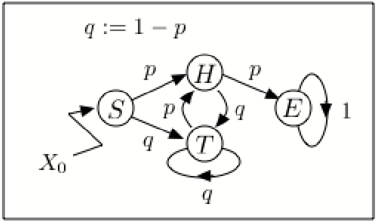

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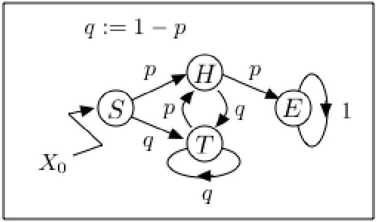

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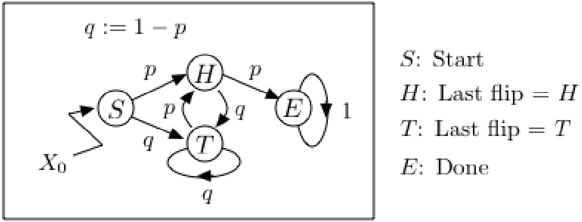

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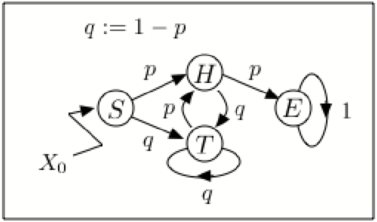

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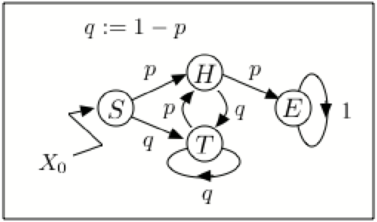

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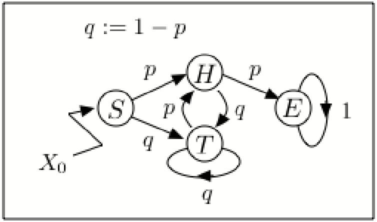

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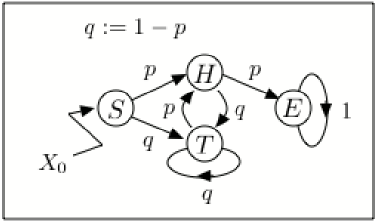

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Solving, we find $\beta(S) = 2 + 3qp^{-1} + q^2p^{-2}$.

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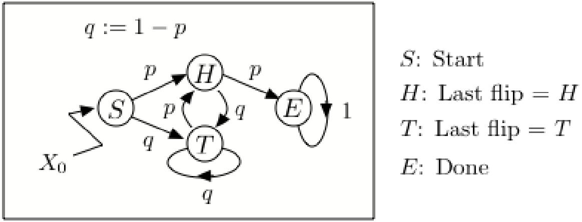

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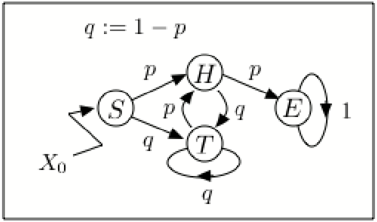
Solving, we find $\beta(S) = 2 + 3qp^{-1} + q^2 p^{-2}$. (E.g., $\beta(S) = 6$ if $p = 1/2$.)

First Passage Time - Example 2



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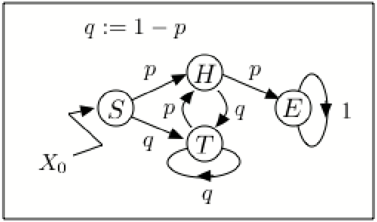
First Passage Time - Example 2



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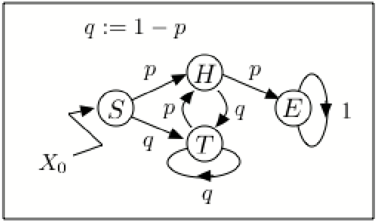
Let us justify the first step equation for $\beta(T)$.

First Passage Time - Example 2



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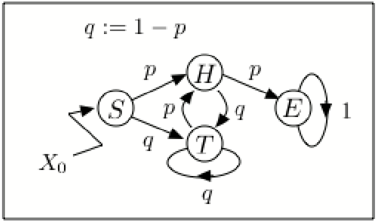
Let us justify the first step equation for $\beta(T)$. The others are similar.

First Passage Time - Example 2



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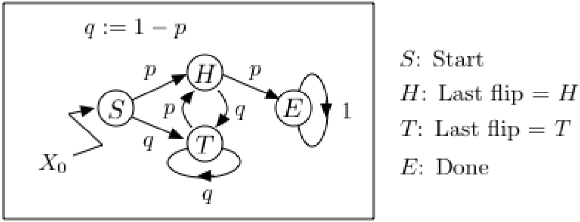
Let us justify the first step equation for $\beta(T)$. The others are similar.

Let $N(T)$ be the random number of steps, starting from T until the MC hits E.

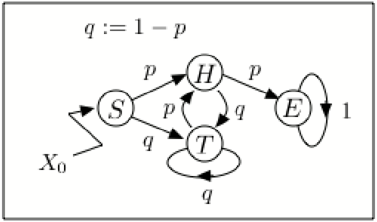
First Passage Time - Example 2



Let us justify the first step equation for $\beta(T)$. The others are similar.

Let $N(T)$ be the random number of steps, starting from T until the MC hits E. Let also $N(H)$ be defined similarly.

First Passage Time - Example 2



S: Start
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Let us justify the first step equation for $\beta(T)$. The others are similar.

Let $N(T)$ be the random number of steps, starting from T until the MC hits E. Let also $N(H)$ be defined similarly. Finally, let $N'(T)$ be the number of steps after the second visit to T until the MC hits E.

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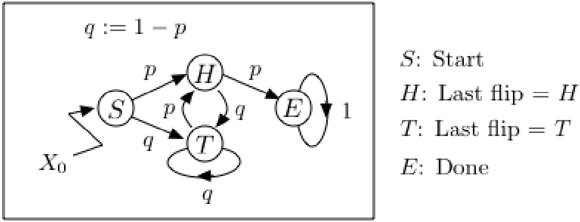

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Let $N(T)$ be the random number of steps, starting from T until the MC hits E. Let also $N(H)$ be defined similarly. Finally, let $N'(T)$ be the number of steps after the second visit to T until the MC hits E. Then,

$$N(T) = 1 + Z \times N(H) + (1 - Z) \times N'(T)$$

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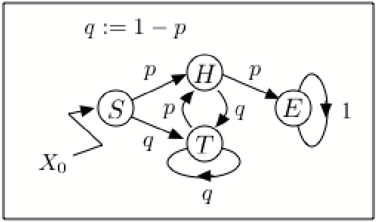

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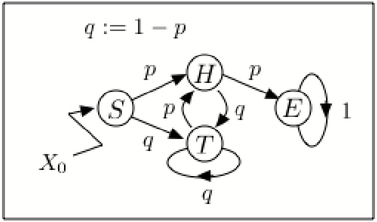

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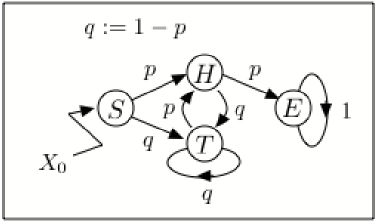

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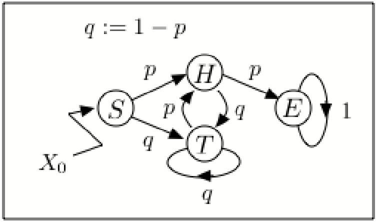

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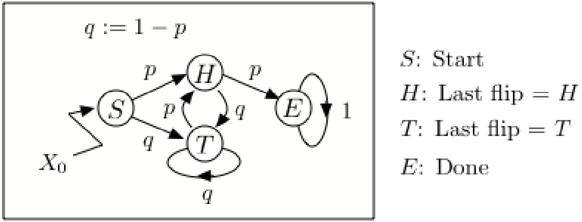

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where $Z = 1\{\text{first flip in } T \text{ is } H\}$. Since Z and $N(H)$ are independent, and Z and $N'(T)$ are independent, taking expectations, we get

$$E[N(T)] = 1 + pE[N(H)] + qE[N'(T)],$$

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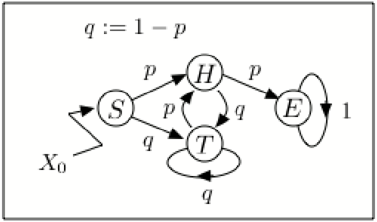

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where $Z = 1\{\text{first flip in } T \text{ is } H\}$. Since Z and $N(H)$ are independent, and Z and $N'(T)$ are independent, taking expectations, we get

$$E[N(T)] = 1 + pE[N(H)] + qE[N'(T)],$$

i.e.,

$$\beta(T) = 1 + p\beta(H) + q\beta(T).$$

First Passage Time - Example 3

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You roll a balanced six-sided die until the sum of the last two rolls is 8.

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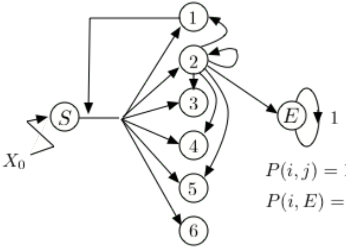

S = Start; E = Done

i = Last roll is i, not done

$P(S, j) = 1/6, j = 1, \ldots, 6$

$P(1, j) = 1/6, j = 1, \ldots, 6$

$P(i, j) = 1/6, i = 2, \ldots, 6; 8 - i \neq j \in \{1, \ldots, 6\}$

$P(i, E) = 1/6, i = 2, \ldots, 6$

The arrows out of $3, \ldots, 6$ (not shown) are similar to those out of 2.

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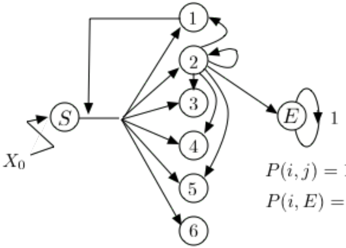

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$P(i, E) = 1/6, i = 2, \ldots, 6$

The arrows out of $3, \ldots, 6$ (not shown) are similar to those out of 2.

$$\beta(S) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j);$$

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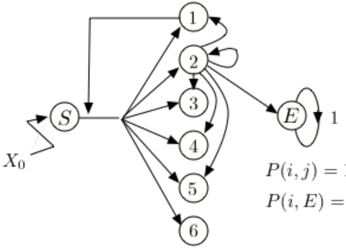

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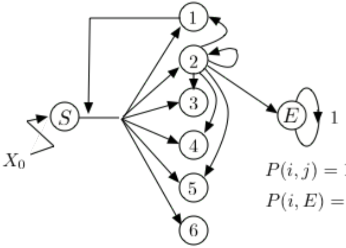

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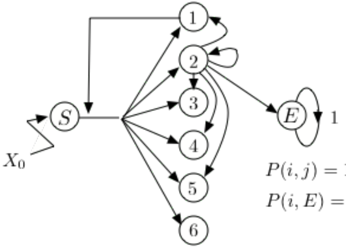

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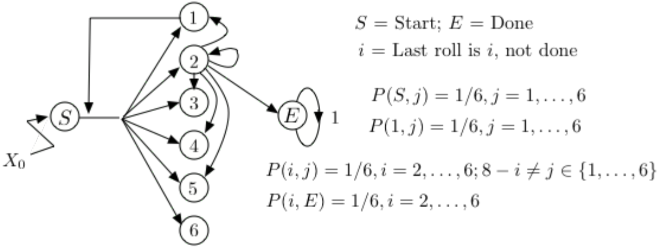

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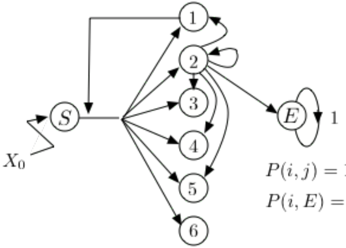

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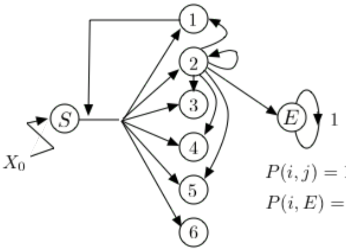

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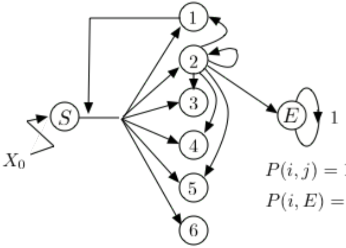

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$$\Rightarrow \cdots \beta(S) = 8.4.$$

First Passage Time - Example 4

You try to go up a ladder that has 20 rungs.

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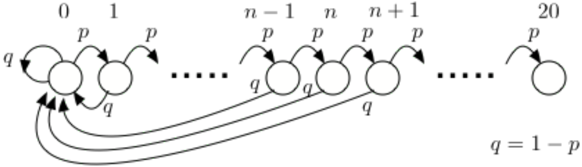

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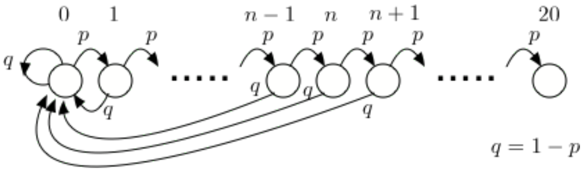

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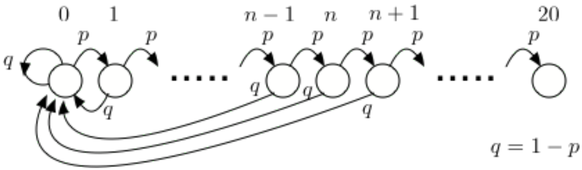

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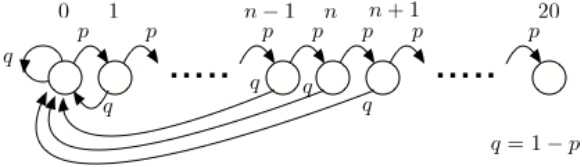

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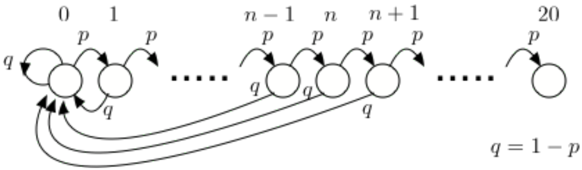

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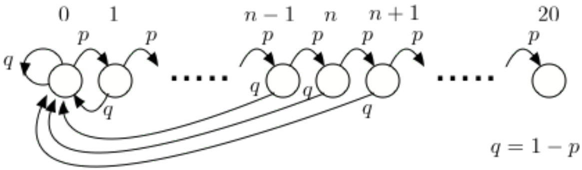

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See Lecture Note 24 for algebra.

First Passage Time - Example 5

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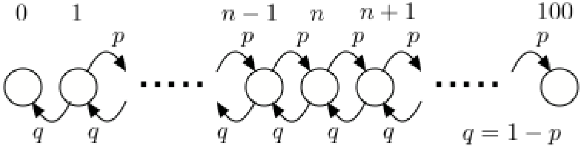

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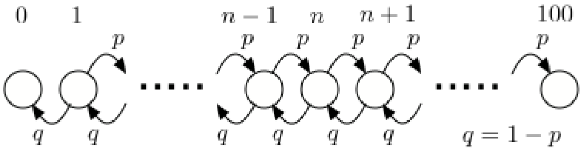

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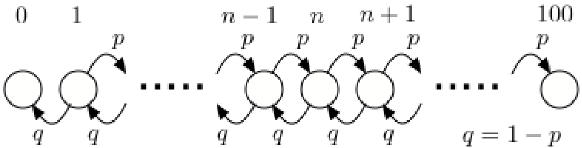

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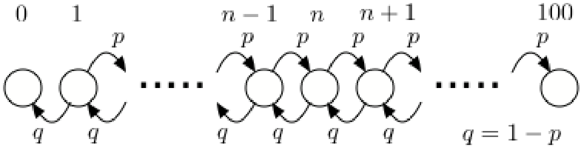

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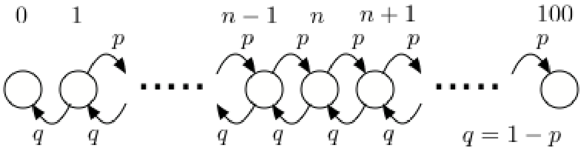

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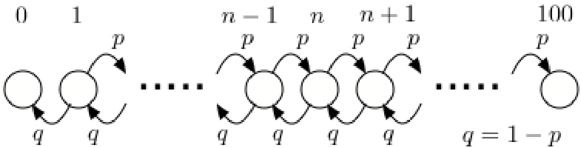

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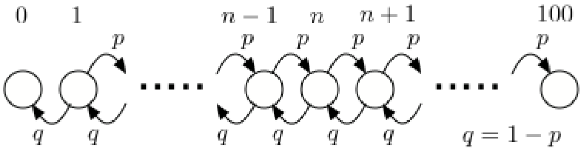

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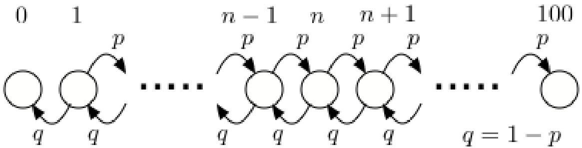

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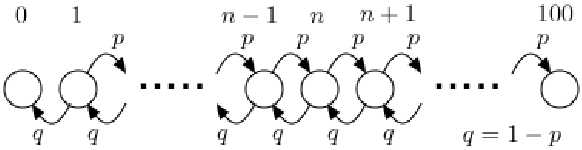

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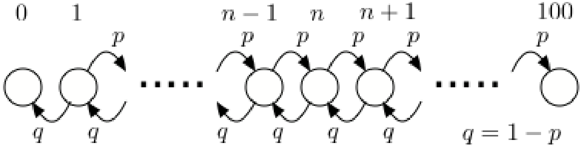

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You play a game of “heads or tails” using a biased coin that yields ‘heads’ with probability 0.48. You start with \$10. At each step, if the flip yields ‘heads’, you earn \$1. Otherwise, you lose \$1. What is the probability that you reach \$100 before \$0?

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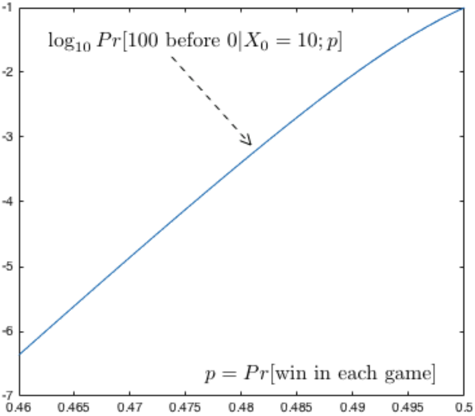

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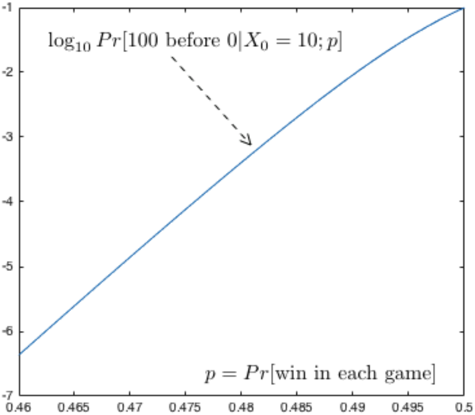

Morale of example:

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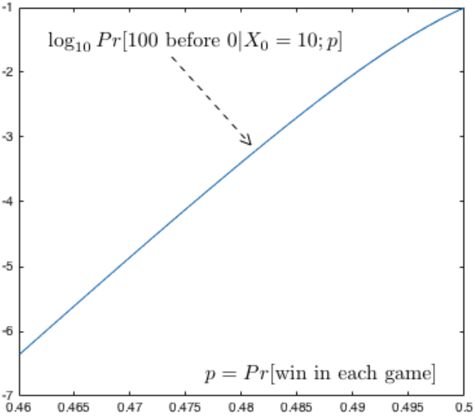

Morale of example: Be careful!

First Step Equations

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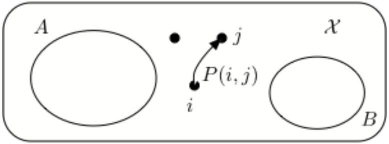

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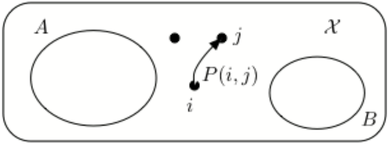

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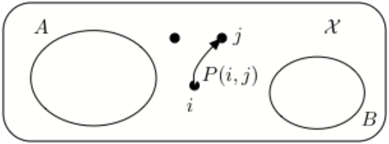

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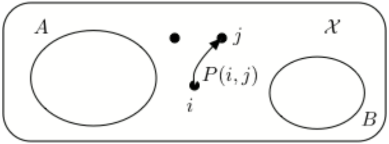

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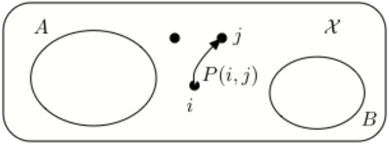

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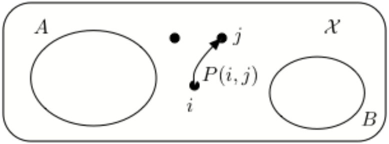

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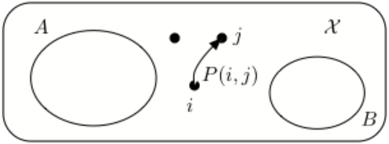

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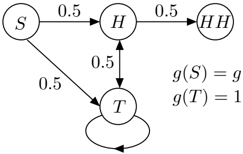

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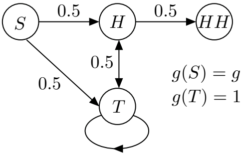

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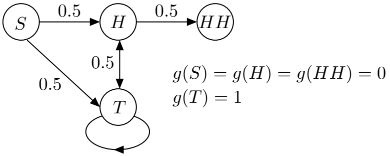

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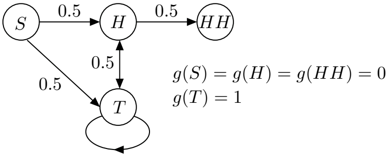

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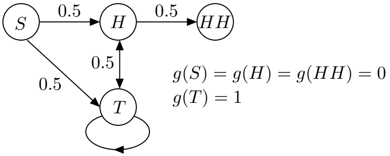

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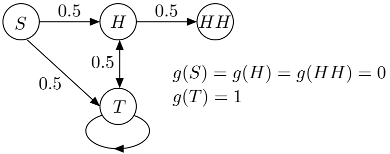

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Solving, we find $\gamma(S) = 2.5$.

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Markov Chains

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1. $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}$
2. $T_A = \min\{n \geq 0 \mid X_n \in A\}$
3. $\alpha(i) = Pr[T_A < T_B | X_0 = i] \Rightarrow FSE$
4. $\beta(i) = E[T_A | X_0 = i] \Rightarrow FSE$
5. $\gamma(i) = E[\sum_{n=0}^{T_A} g(X_n) | X_0 = i] \Rightarrow FSE$.