CS70: Jean Walrand: Lecture 33.

Markov Chains 2

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Markov Chains 2

1. Review
2. Distribution
3. Irreducibility
4. Convergence

Review

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- Markov Chain:

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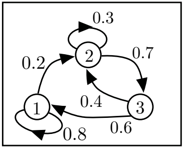
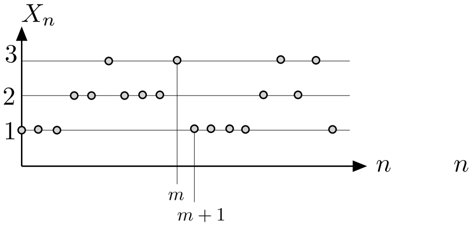
Distribution of X_n

Distribution of X_n





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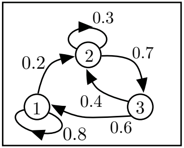

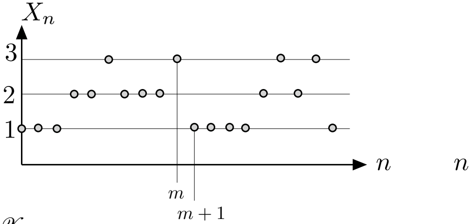

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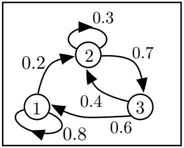

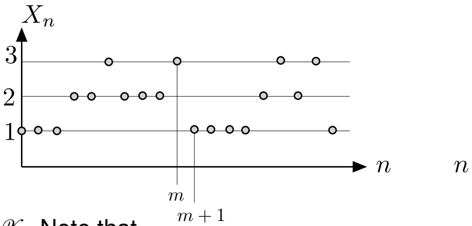

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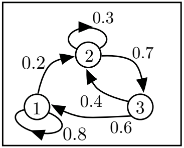

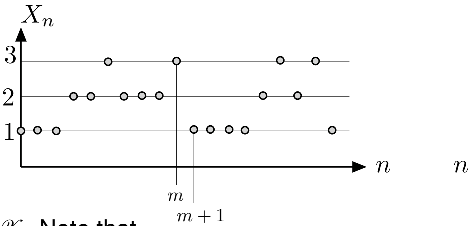

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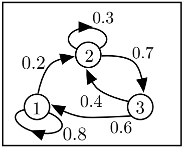

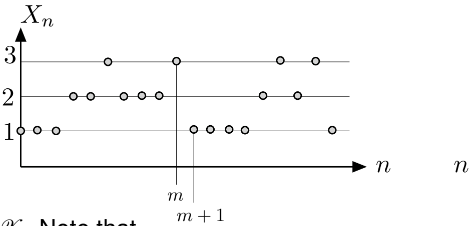

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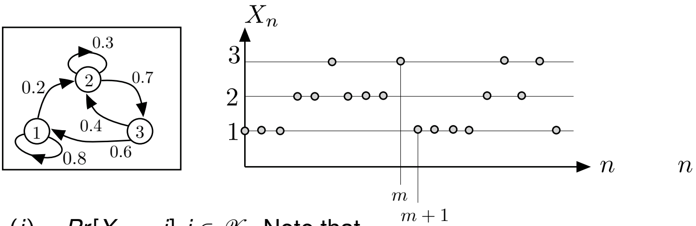

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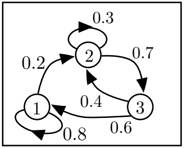

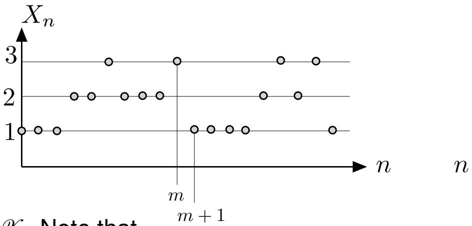

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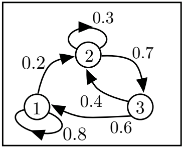

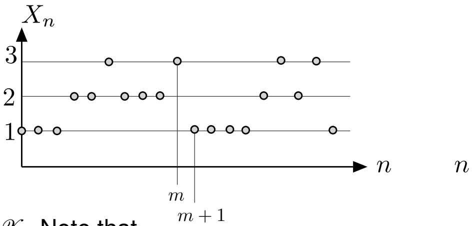

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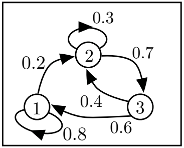

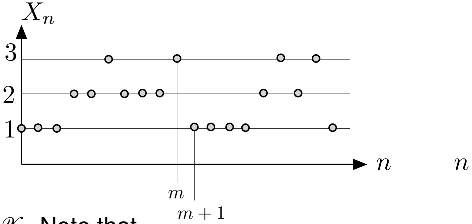

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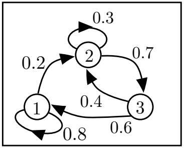

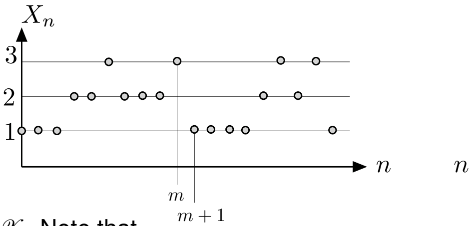

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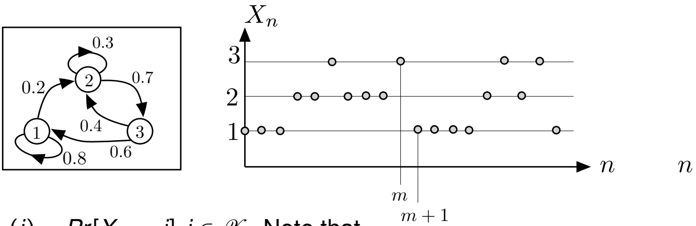

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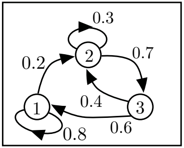

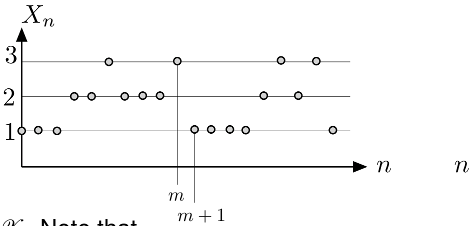

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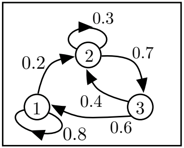

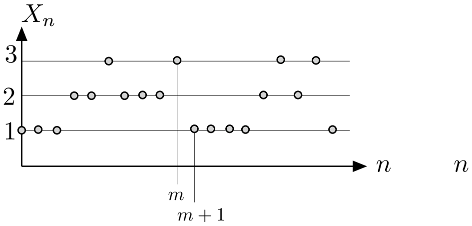

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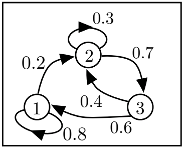

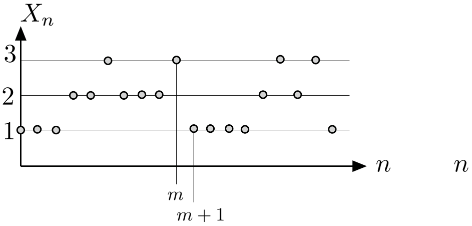

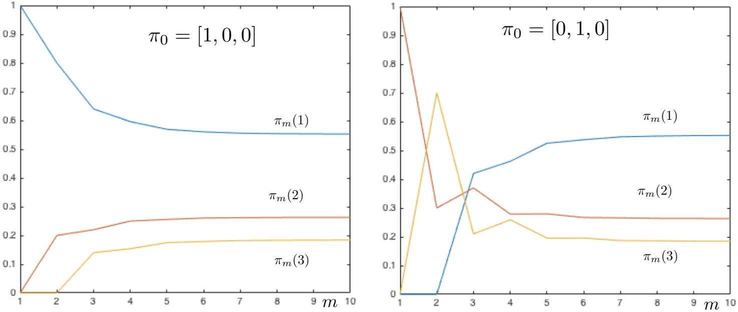

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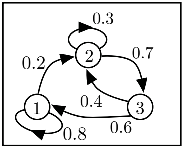

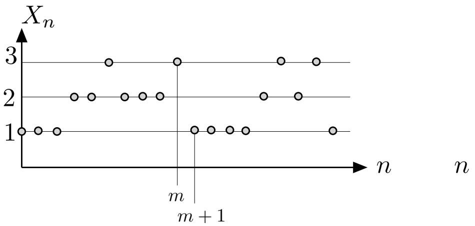

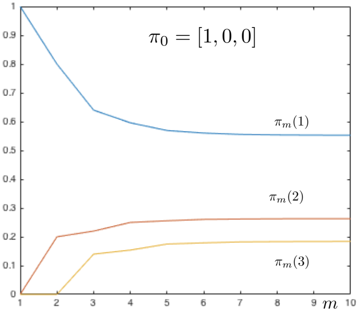

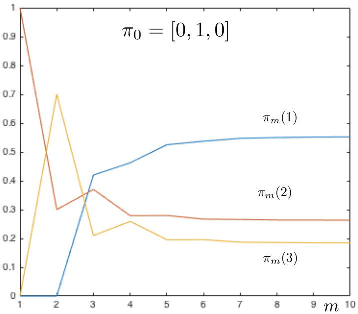

As m increases, π_m converges to a vector that does not depend on π_0.

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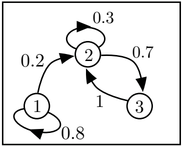

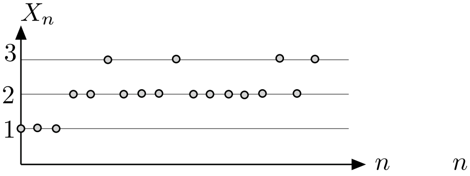

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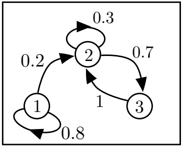

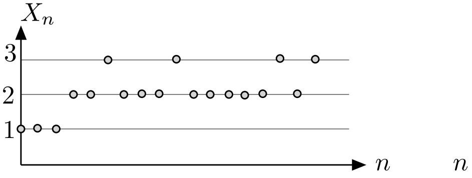

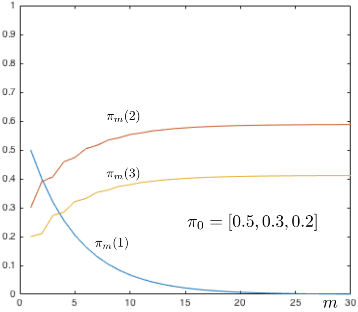

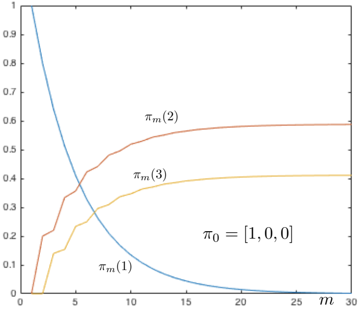

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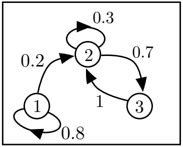

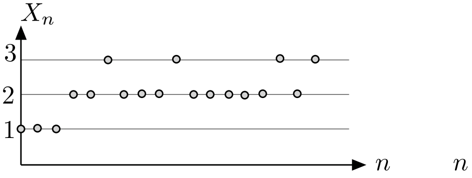

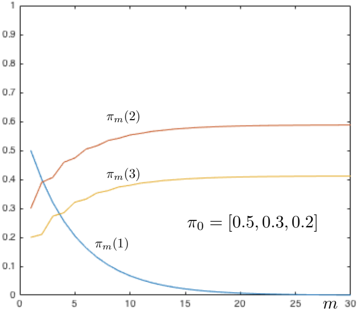

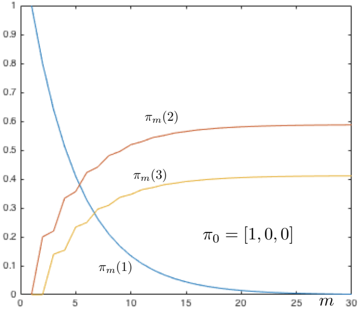

As m increases, π_m converges to a vector that does not depend on π_0.

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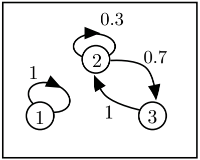

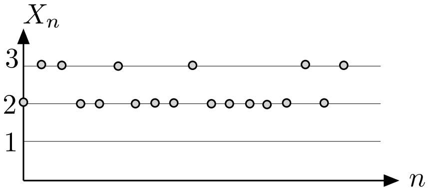

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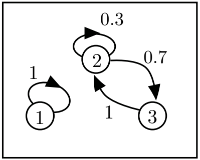

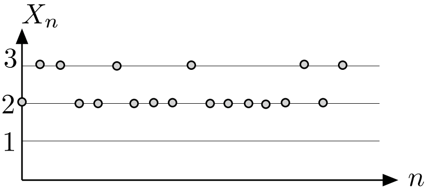

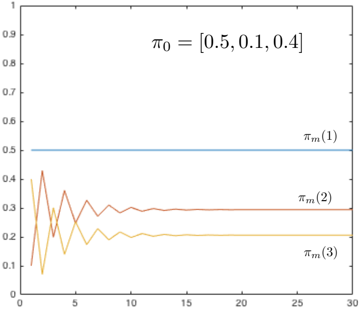

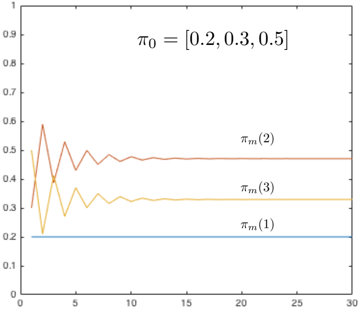

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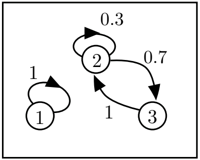

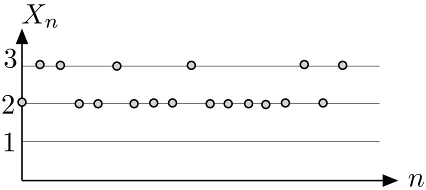

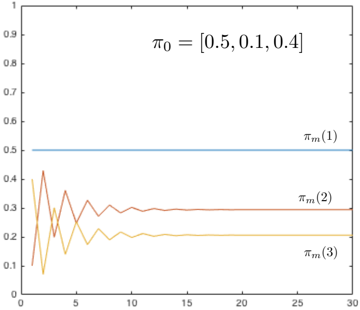

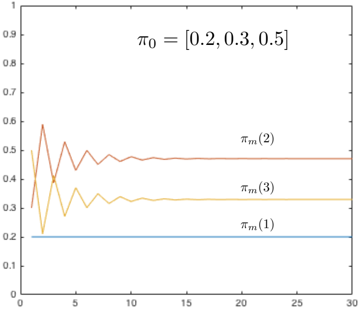

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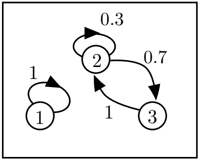

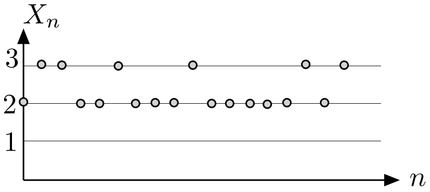

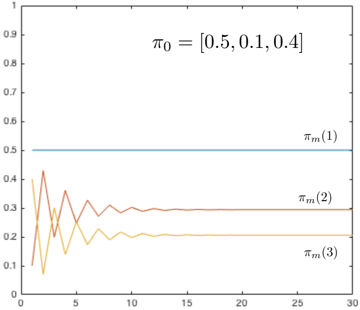

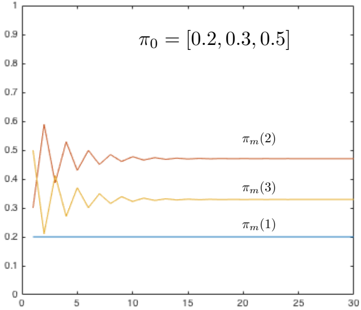

As m increases, π_m converges to a vector that depends on π_0 (obviously, since $\pi_m(1) = \pi_0(1), \forall m$).

Balance Equations

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Thus, $Pr[\text{enter } i] = Pr[\text{leave } i]$.

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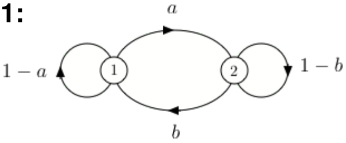

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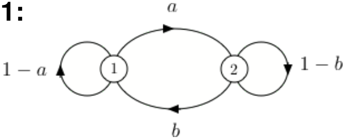

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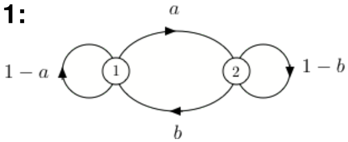

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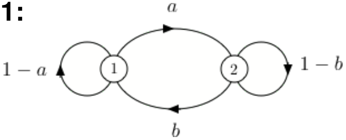

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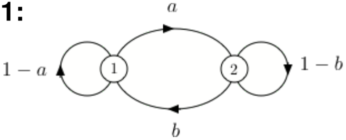

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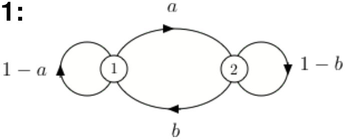

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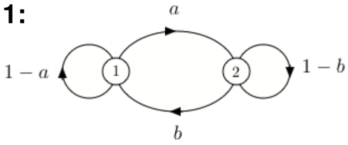

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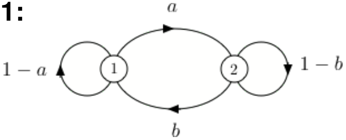

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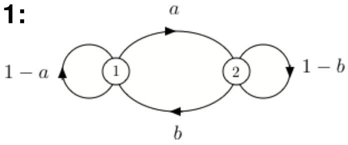

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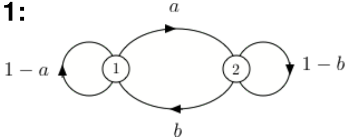

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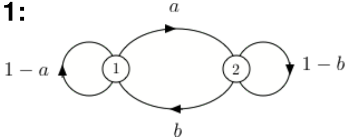

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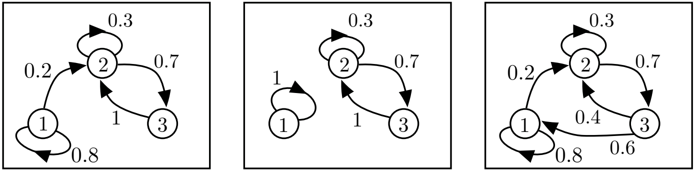

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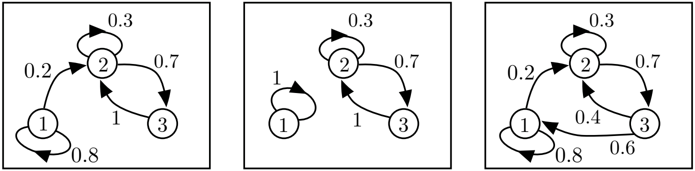

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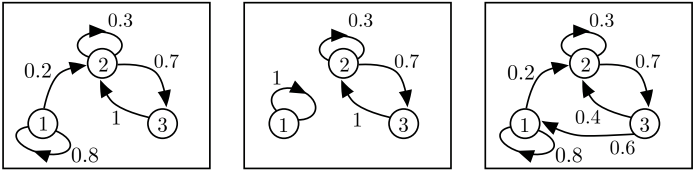

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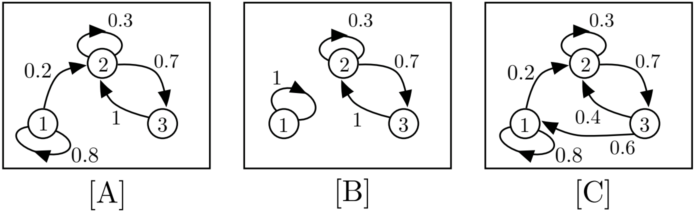

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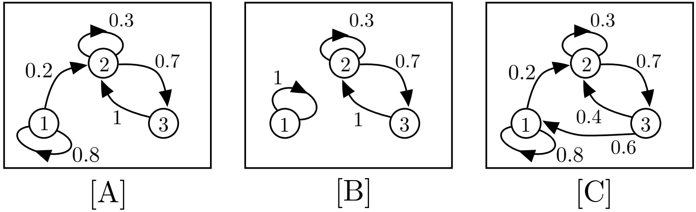

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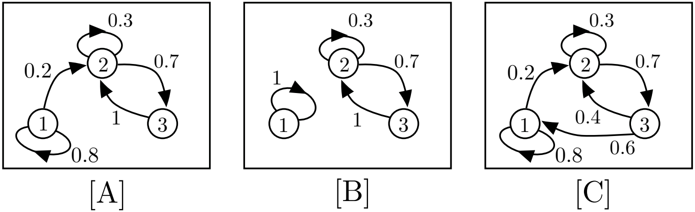

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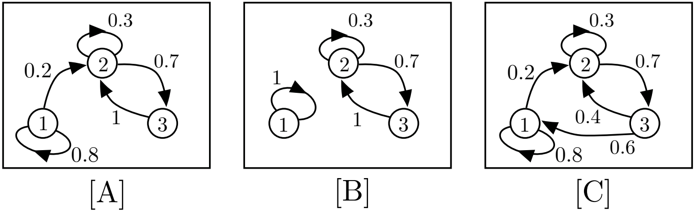

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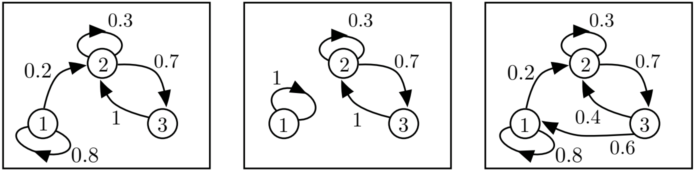

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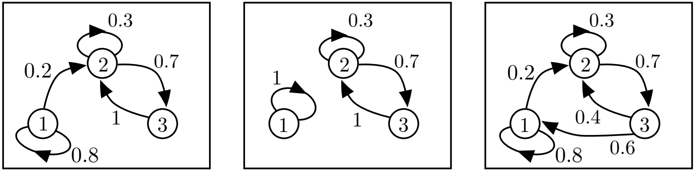

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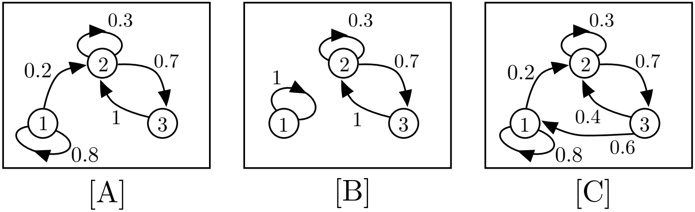

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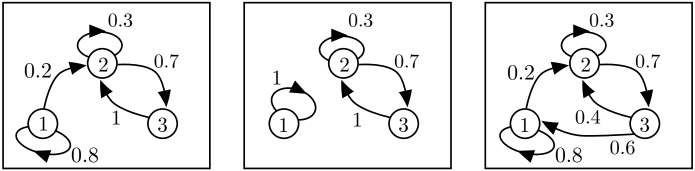

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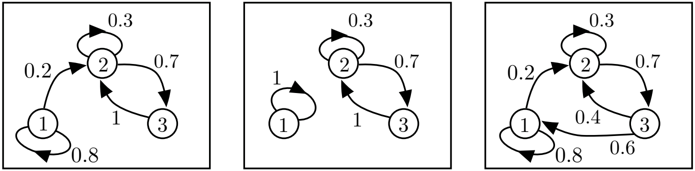

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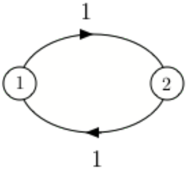

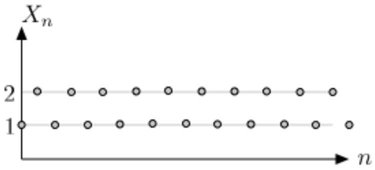

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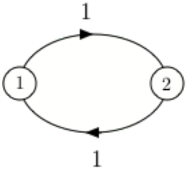

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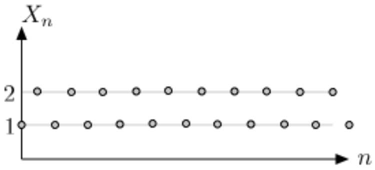

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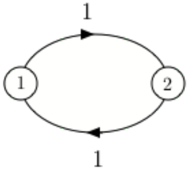

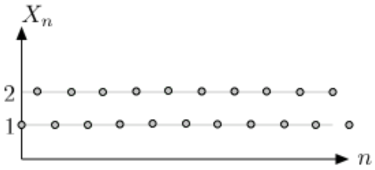

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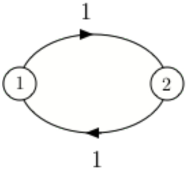

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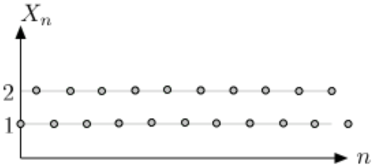

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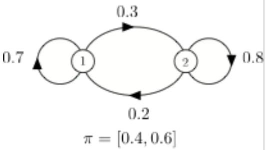

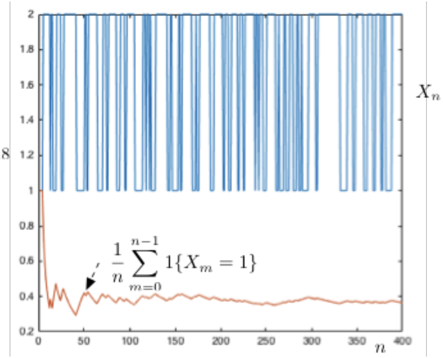

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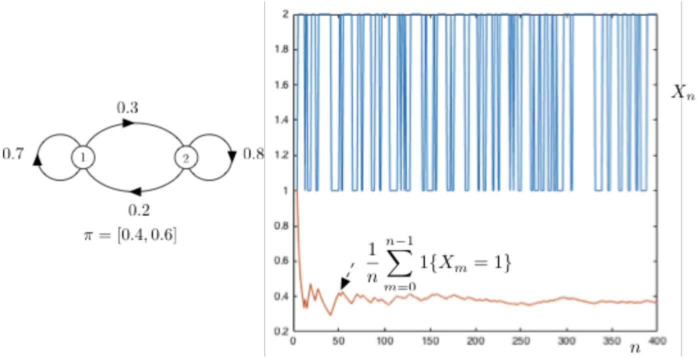

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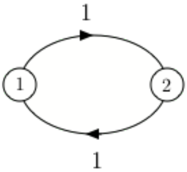

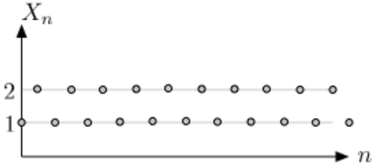

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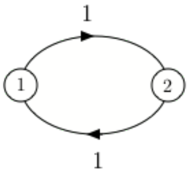

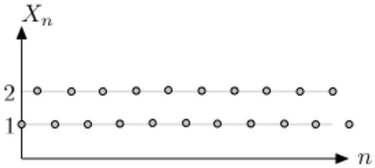

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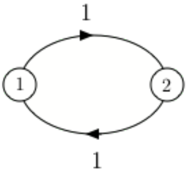

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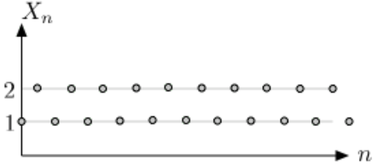

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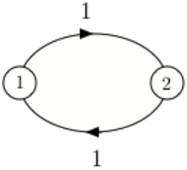

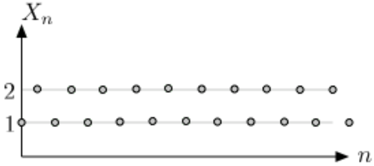

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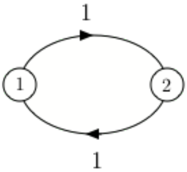

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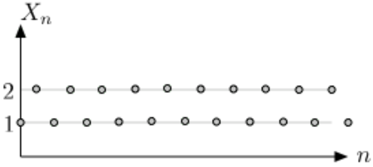

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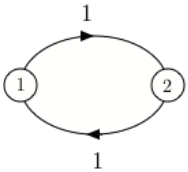

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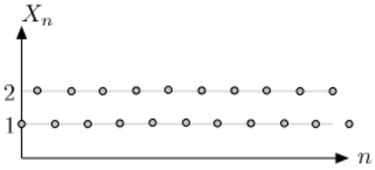

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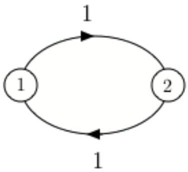

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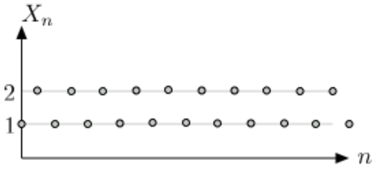

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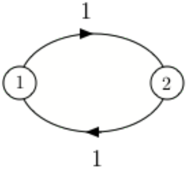

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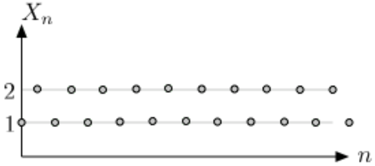

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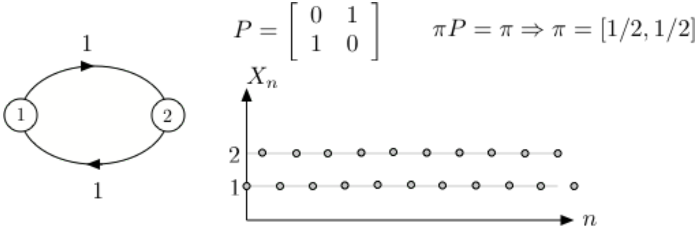

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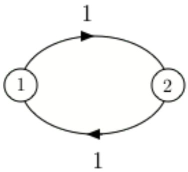

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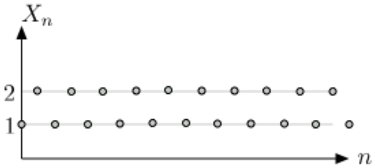

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Thus, if $\pi_0 = [1, 0]$, $\pi_1 = [0, 1], \pi_2 = [1, 0], \pi_3 = [0, 1]$, etc.

Hence, π_n does not converge to $\pi = [1/2, 1/2]$.

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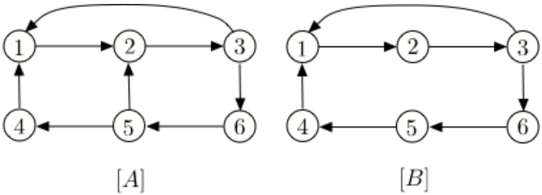

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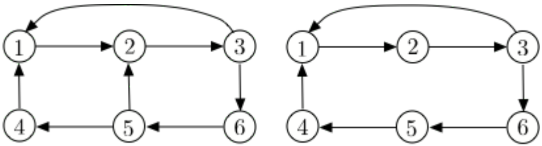

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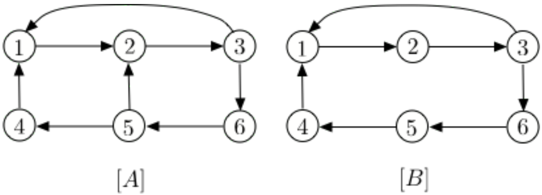

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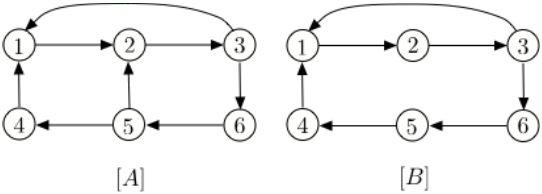

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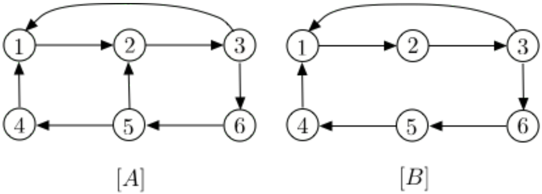

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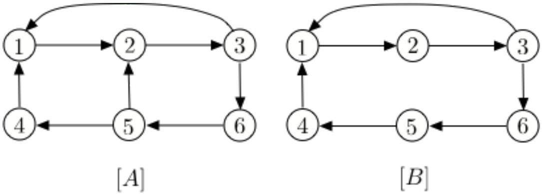

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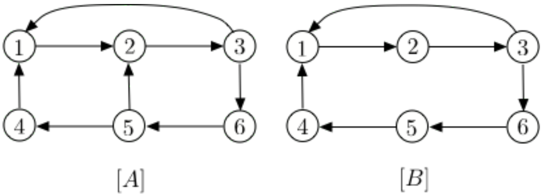

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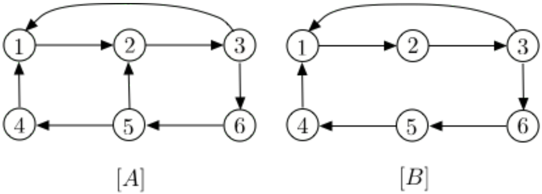

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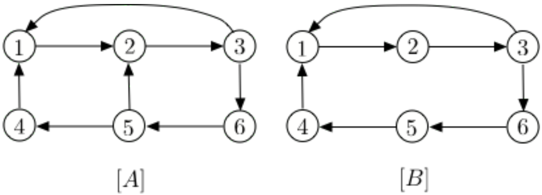

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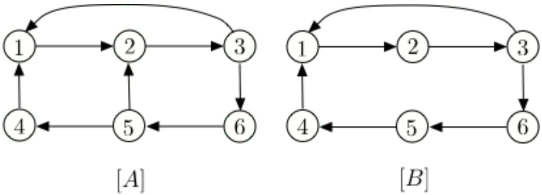

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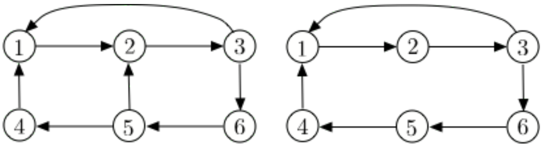

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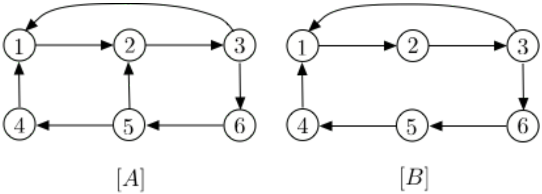

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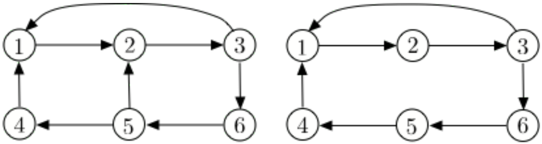

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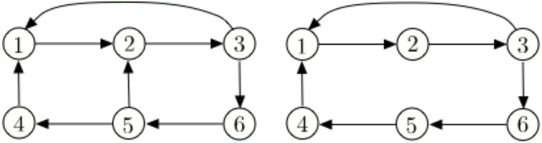

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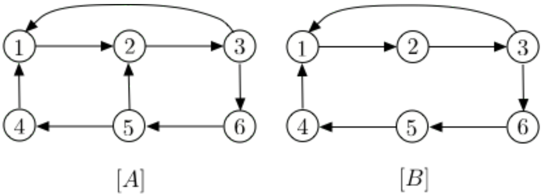

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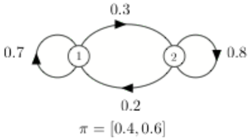

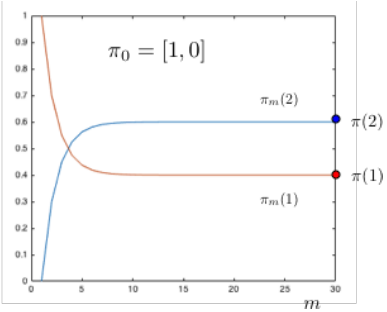

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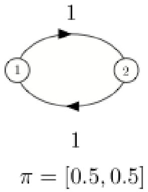

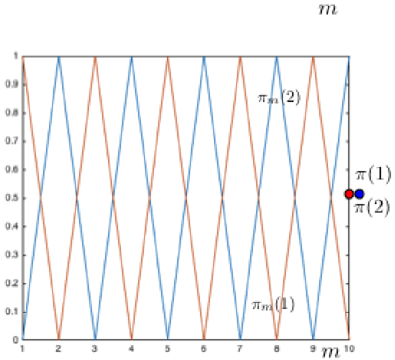

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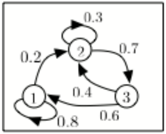

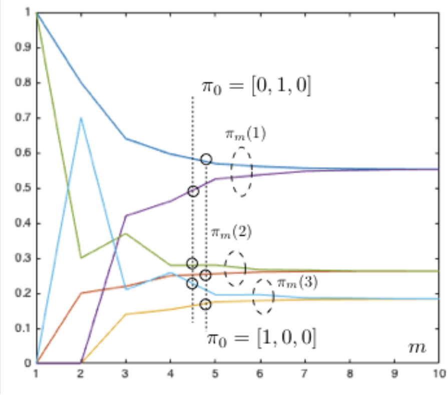

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