CS70: Jean Walrand: Lecture 34.

Continuous Probability 1

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1. Examples
2. Events
3. Continuous Random Variables

Uniformly at Random in $[0, 1]$.

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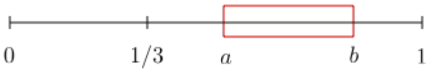

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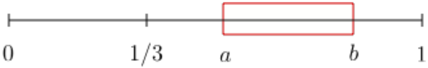

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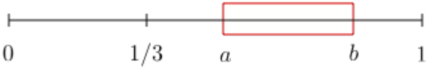

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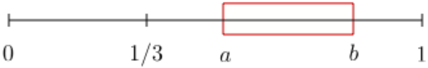

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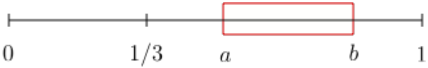

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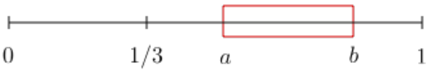

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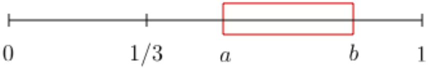

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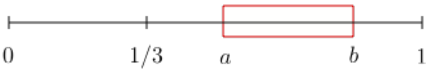

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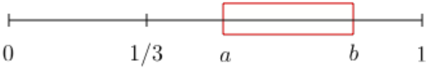

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Makes sense: $b - a$ is the fraction of $[0,1]$ that $[a,b]$ covers.

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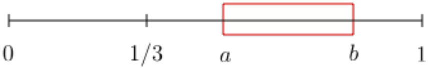

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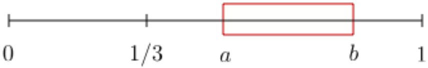

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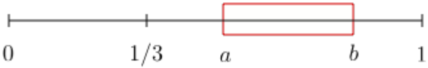

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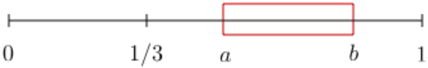

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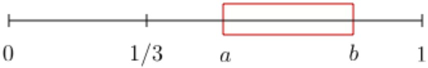

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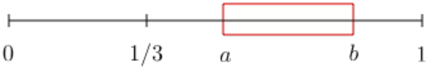

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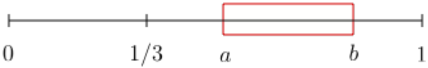

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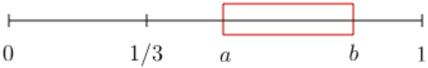

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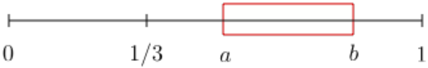

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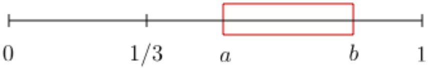

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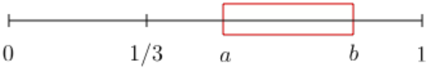

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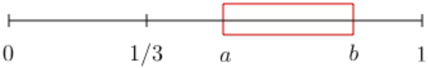

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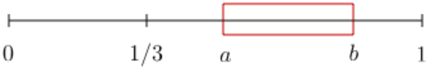

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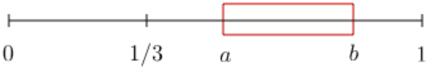

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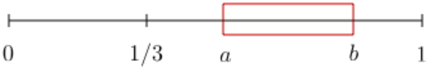

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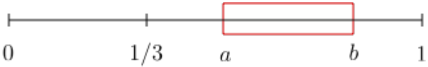

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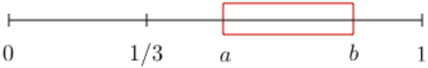

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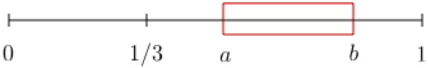

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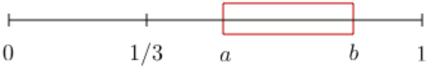

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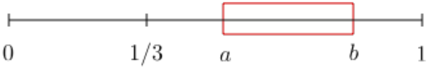

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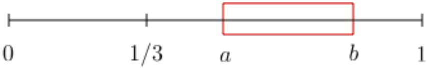

Note: $Pr[X \leq x] = x$ for $x \in [0, 1]$. Also, $Pr[X \leq x] = 0$ for $x < 0$ and $Pr[X \leq x] = 1$ for $x > 1$. Let us define $F(x) = Pr[X \leq x]$.

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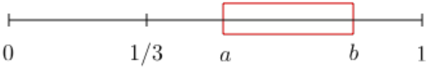

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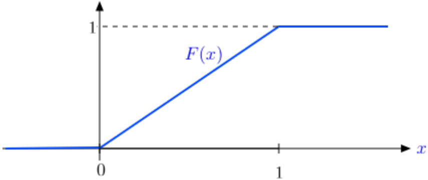

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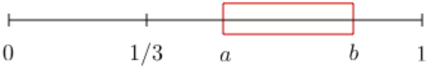

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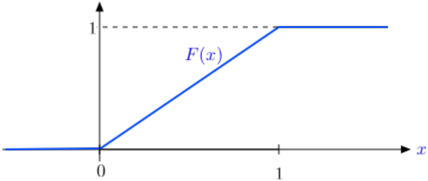
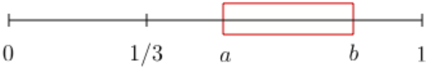


Then we have $Pr[X \in (a, b]] = Pr[X \leq b] - Pr[X \leq a]$

Uniformly at Random in $[0,1]$.



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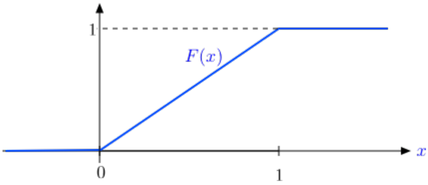

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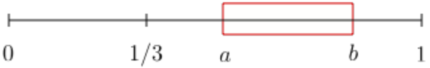
Uniformly at Random in $[0, 1]$.



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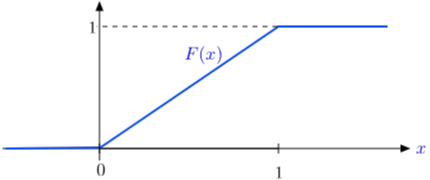
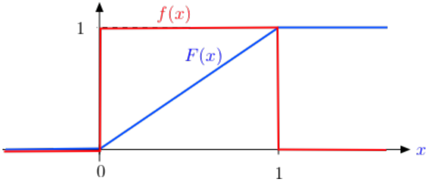


Then we have $Pr[X \in (a, b]] = Pr[X \leq b] - Pr[X \leq a] = F(b) - F(a)$.

Thus, $F(\cdot)$ specifies the probability of all the events!

Uniformly at Random in $[0,1]$.



$$Pr[X \in (a,b]] = Pr[X \le b] - Pr[X \le a]$$

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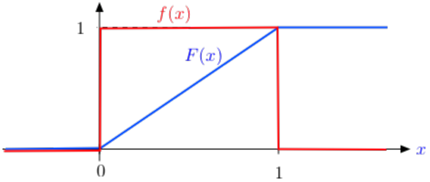

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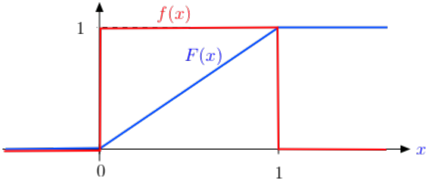
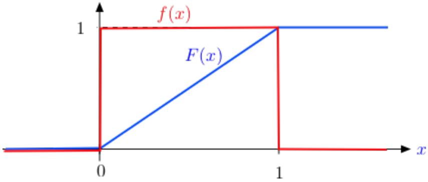


$$Pr[X \in (a, b]] = Pr[X \le b] - Pr[X \le a] = F(b) - F(a).$$

An alternative view is to define $f(x) = \frac{d}{dx}F(x) =$

Uniformly at Random in $[0, 1]$.



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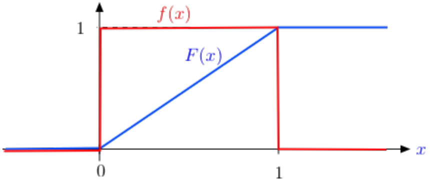
Uniformly at Random in $[0,1]$.



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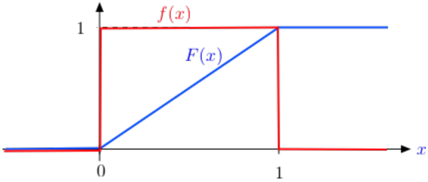
Uniformly at Random in $[0, 1]$.



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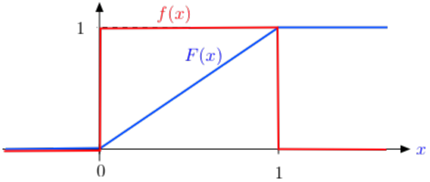
Thus, the probability of an event is the integral of $f(x)$ over the event:

Uniformly at Random in $[0, 1]$.



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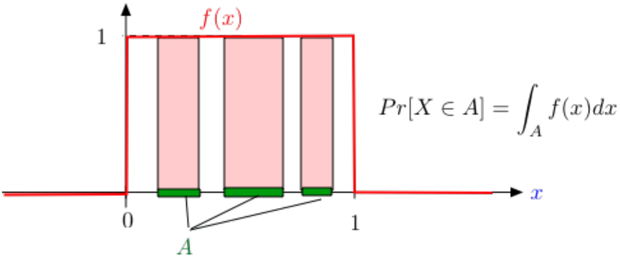
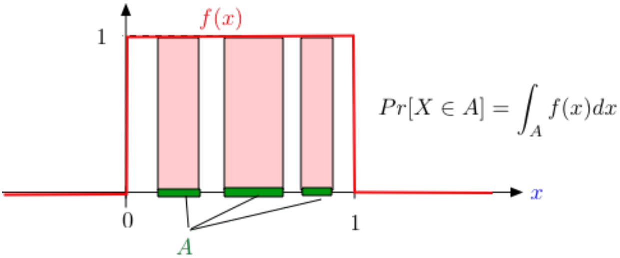
Thus, the probability of an event is the integral of $f(x)$ over the event:

$$Pr[X \in A] = \int_A f(x)dx.$$

Uniformly at Random in $[0, 1]$.

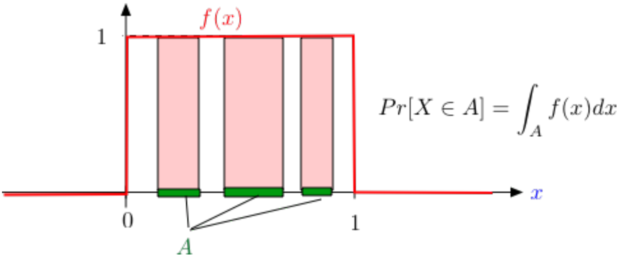


Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
one unit of probability is spread over $[0, 1]$:

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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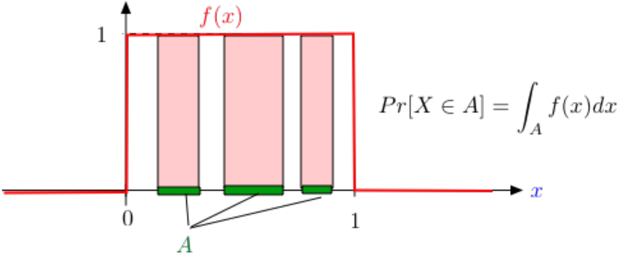

Think of $f(x)$ as describing how
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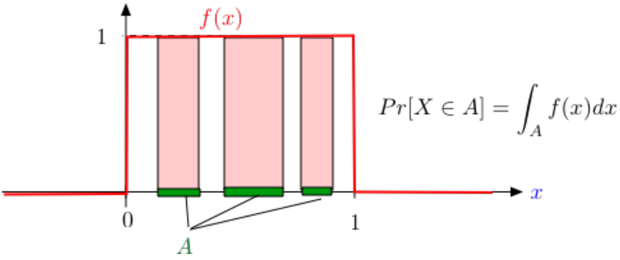
Then $Pr[X \in A]$ is the probability mass over A.

Uniformly at Random in $[0,1]$.



Think of $f(x)$ as describing how
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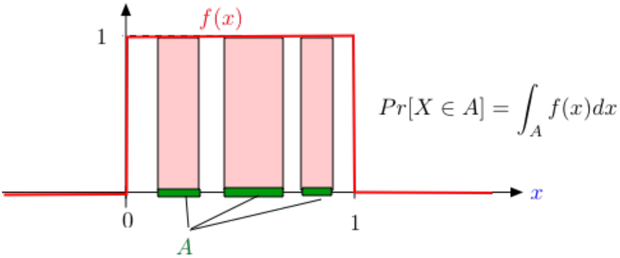
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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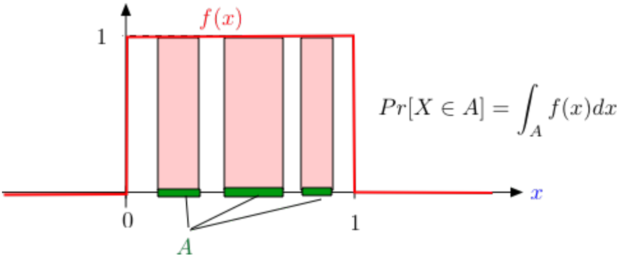
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

- This makes the probability automatically additive.

Uniformly at Random in $[0,1]$.



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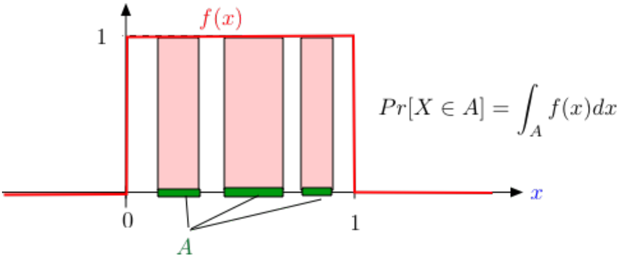
- We need $f(x) \geq 0$

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
one unit of probability is spread over $[0, 1]$: uniformly!

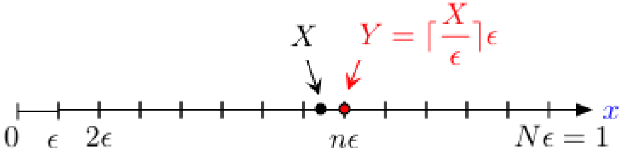
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

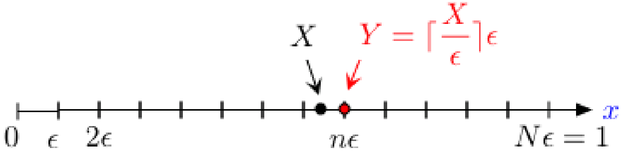
- This makes the probability automatically additive.
- We need $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Uniformly at Random in $[0, 1]$.

Uniformly at Random in $[0, 1]$.

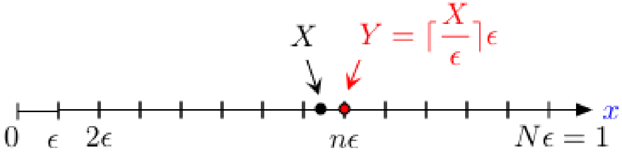


Uniformly at Random in $[0, 1]$.



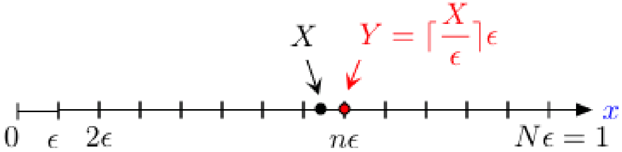
Discrete Approximation:

Uniformly at Random in $[0, 1]$.



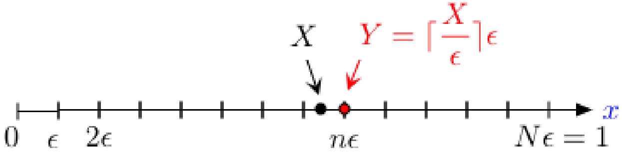
Discrete Approximation: Fix $N \gg 1$

Uniformly at Random in $[0,1]$.



Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

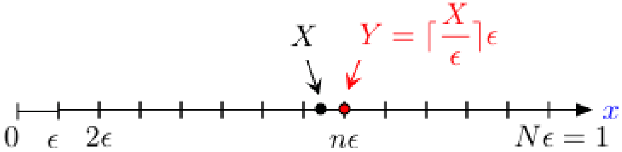
Uniformly at Random in $[0,1]$.



Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Uniformly at Random in $[0,1]$.

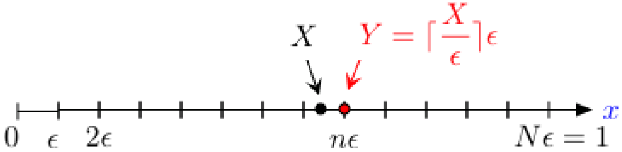


Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Then $|X - Y| \leq \varepsilon$

Uniformly at Random in $[0, 1]$.

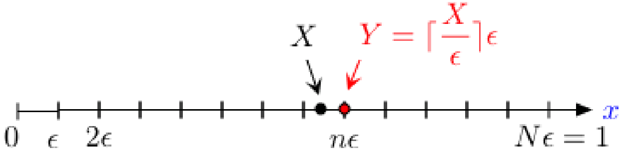


Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Then $|X - Y| \leq \varepsilon$ and Y is discrete:

Uniformly at Random in $[0,1]$.



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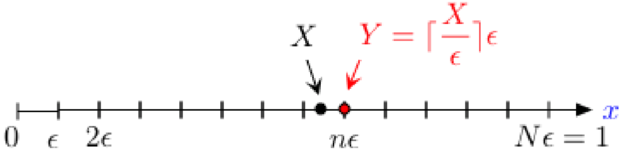
Then $|X - Y| \leq \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Uniformly at Random in $[0, 1]$.



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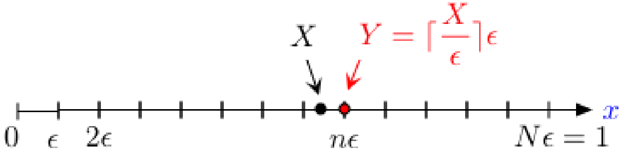
Then $|X - Y| \leq \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

Uniformly at Random in $[0,1]$.



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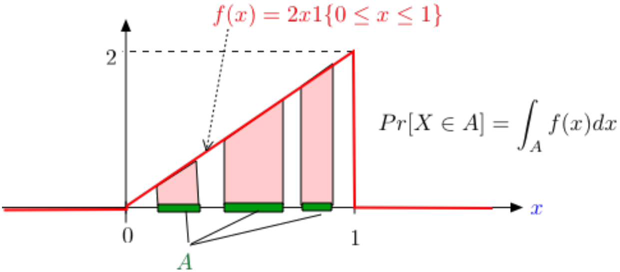
Then $|X - Y| \leq \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

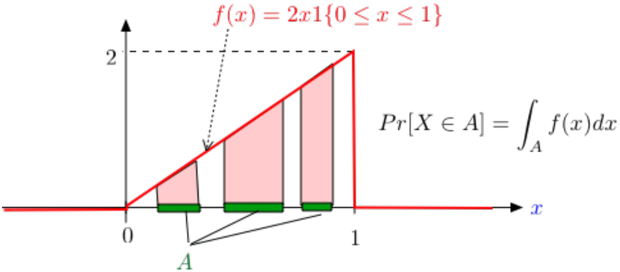
Thus, X is 'almost discrete.'

Nonuniformly at Random in $[0, 1]$.

Nonuniformly at Random in [0, 1].

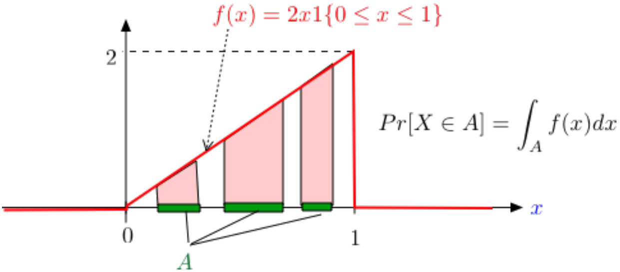


Nonuniformly at Random in $[0, 1]$.



This figure shows a different choice of $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x)dx = 1$.

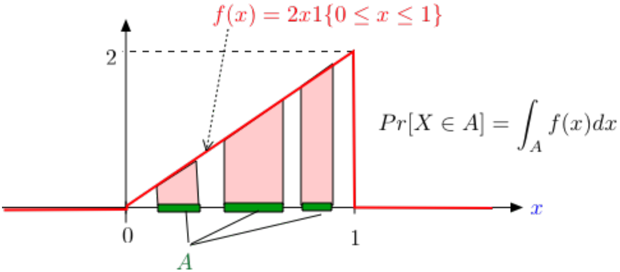
Nonuniformly at Random in $[0, 1]$.



This figure shows a different choice of $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x)dx = 1$.

It defines another way of choosing X at random in $[0, 1]$.

Nonuniformly at Random in $[0, 1]$.



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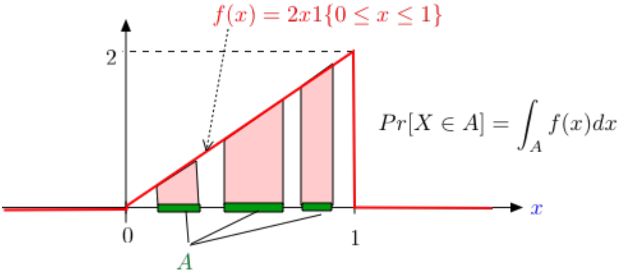
Note that X is more likely to be closer to 1 than to 0.

Nonuniformly at Random in $[0, 1]$.



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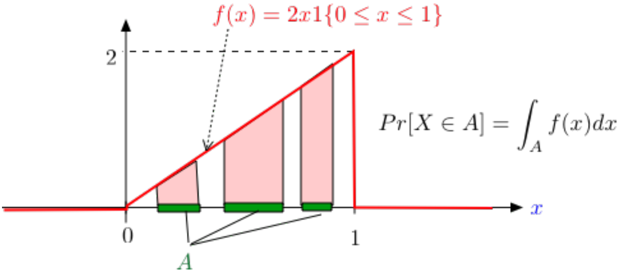
Nonuniformly at Random in $[0, 1]$.



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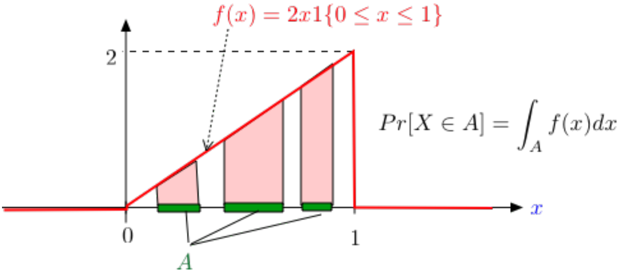
Nonuniformly at Random in $[0, 1]$.



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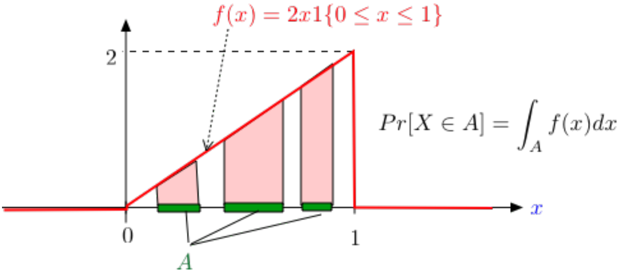
Nonuniformly at Random in $[0, 1]$.



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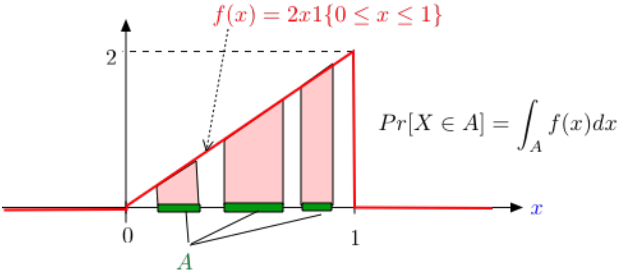
One has $Pr[X \leq x] = \int_{-\infty}^{x} f(u)du = x^2$ for $x \in [0, 1]$.

Also, $Pr[X \in (x, x+\varepsilon)] = \int_{x}^{x+\varepsilon} f(u)du$

Nonuniformly at Random in $[0,1]$.



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Another Nonuniform Choice at Random in $[0, 1]$.

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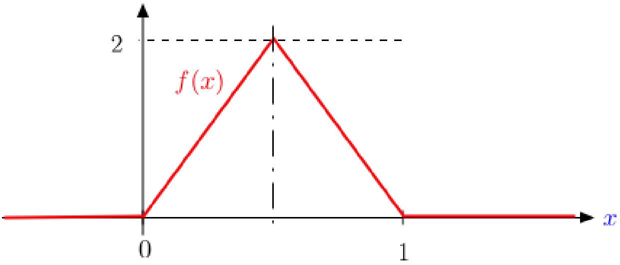

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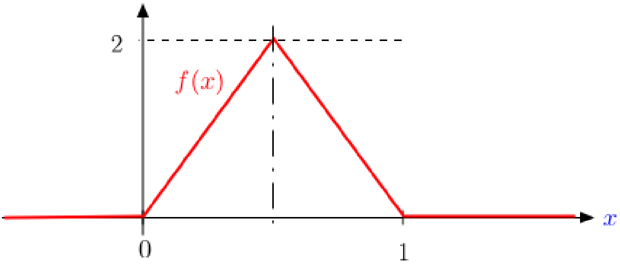

This figure shows yet a different choice of $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x)dx = 1$.

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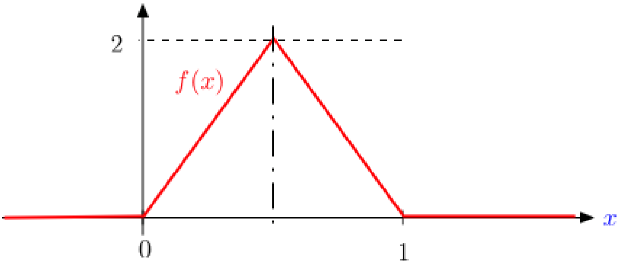

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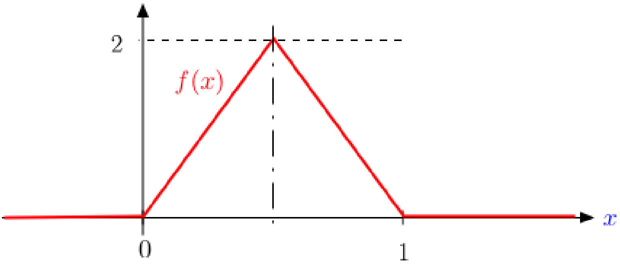

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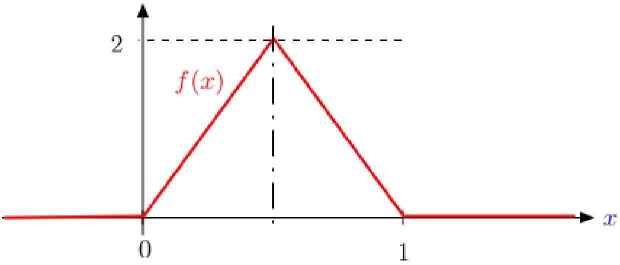

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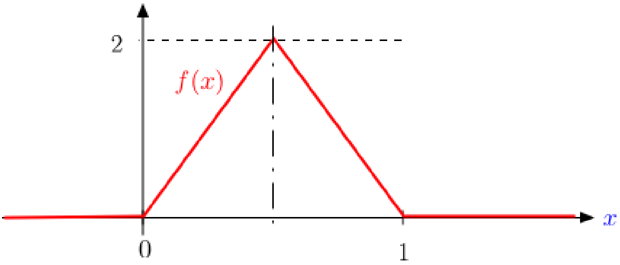

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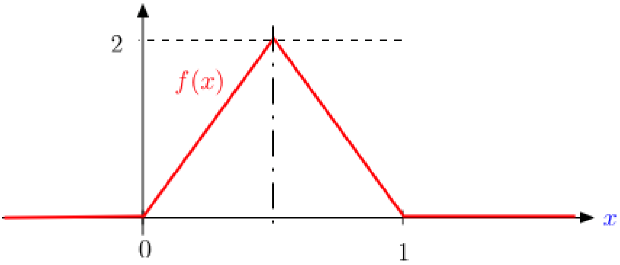

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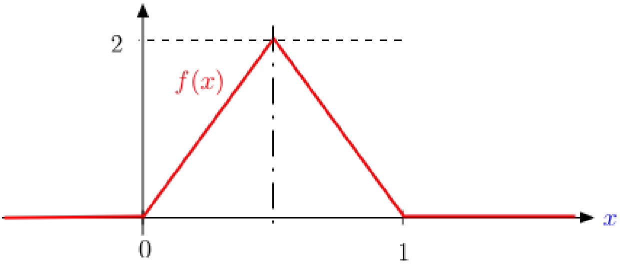

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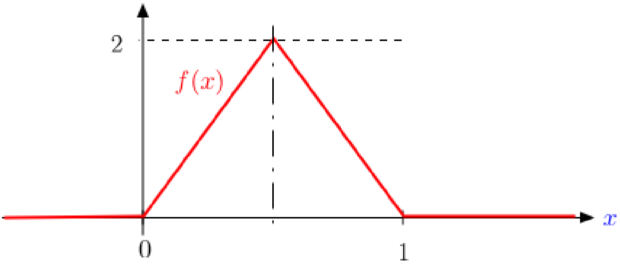

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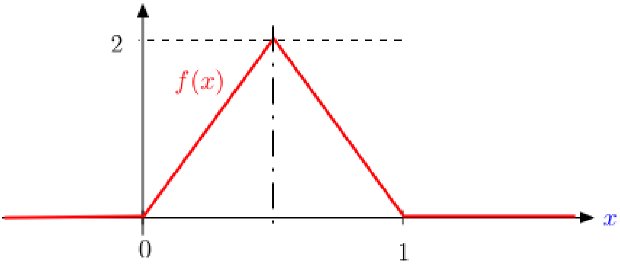

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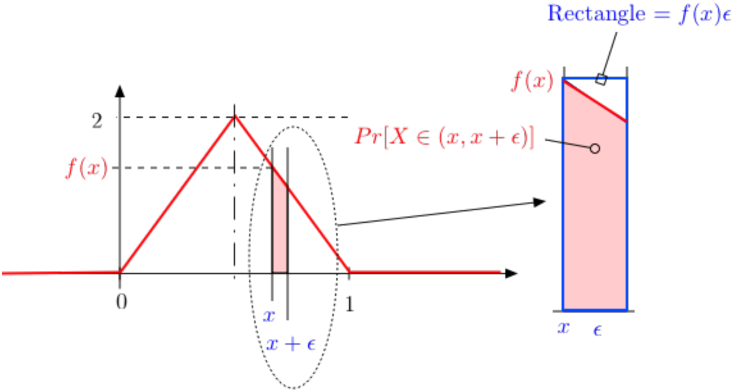

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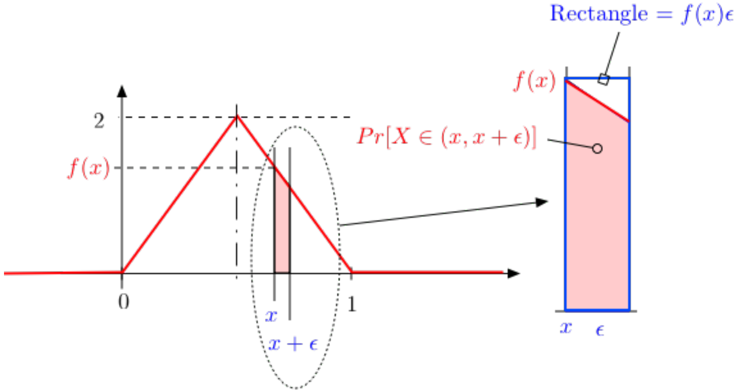

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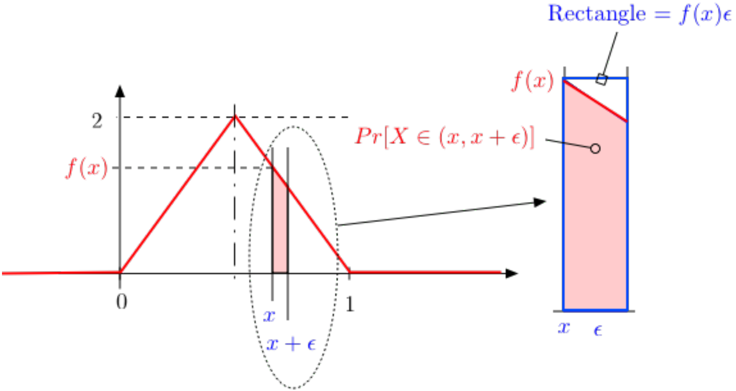

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Example: hitting random location on gas tank.

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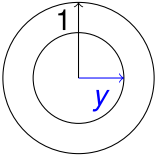

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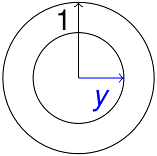

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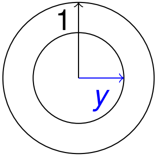

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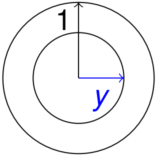

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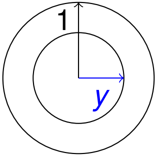

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Calculation of event with dartboard..

Probability between .5 and .6 of center?

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Use whichever is convenient.

Target

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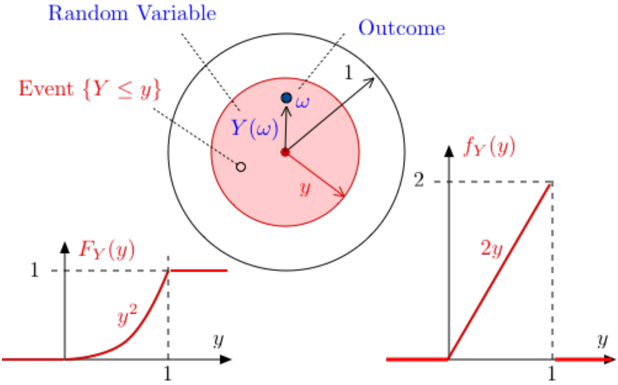

Random Variable
Outcome
Event $\{Y \leq y\}$
1
ω
$Y(\omega)$
y
$F_Y(y)$
1
y^2
y
1
$f_Y(y)$
2
$2y$
y
1

$U[a, b]$

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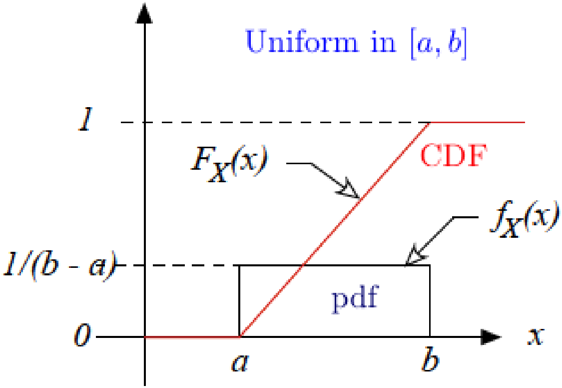

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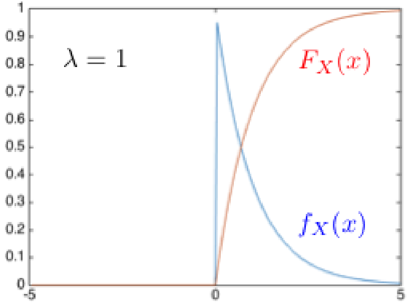

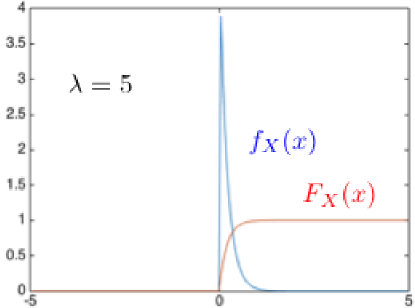

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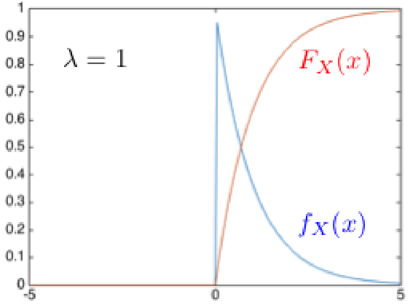

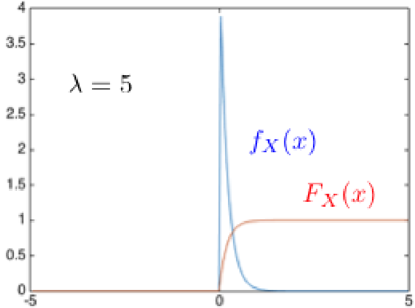

Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

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Continuous random variable X, specified by

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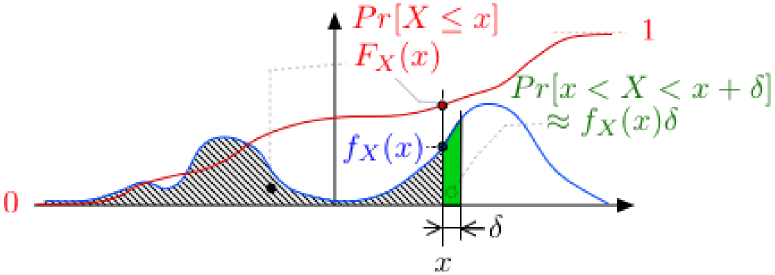
Recall that $Pr[X \in (x, x+\delta)] \approx f_X(x)\delta$. Think of X taking discrete values $n\delta$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$ with $Pr[X = n\delta] = f_X(n\delta)\delta$.

A Picture



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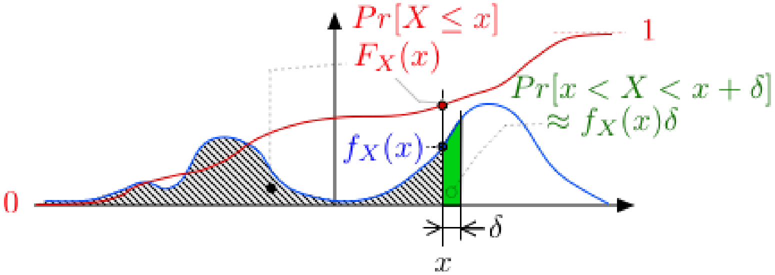

The pdf $f_X(x)$ is a nonnegative function that integrates to 1.

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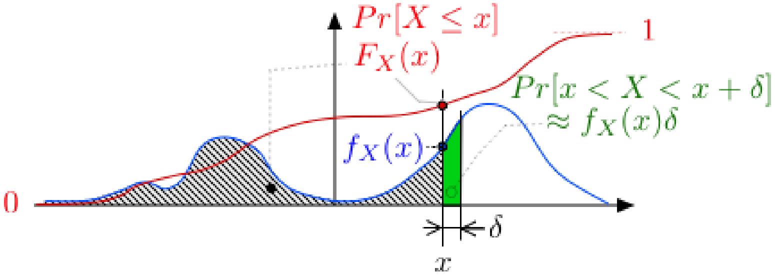

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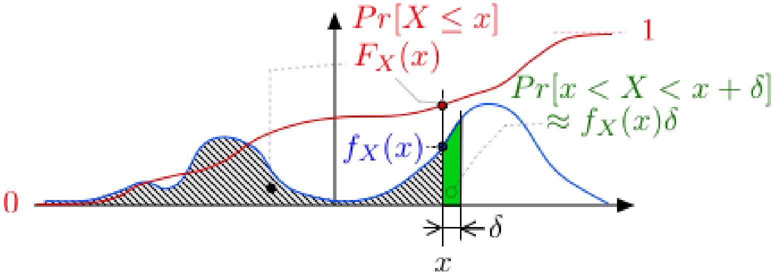

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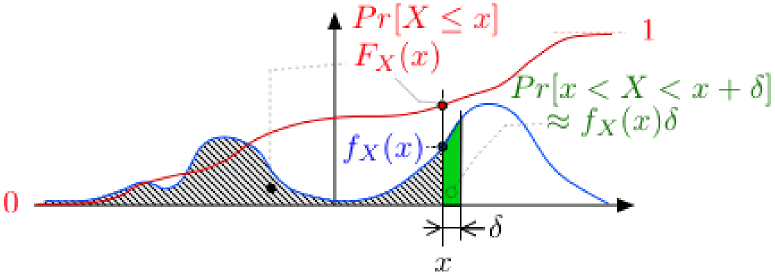

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Summary

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