CS70: Jean Walrand: Lecture 36.

Continuous Probability 3

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1. Review: CDF, PDF
2. Review: Expectation
3. Review: Independence
4. Meeting at a Restaurant
5. Breaking a Stick
6. Maximum of Exponentials
7. Quantization Noise
8. Replacing Light Bulbs
9. Expected Squared Distance
10. Geometric and Exponential

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Justifications: Think of the discrete approximations of the continuous RVs.

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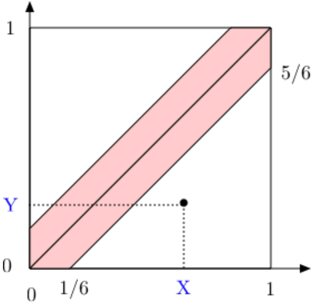

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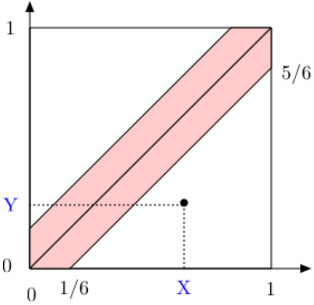

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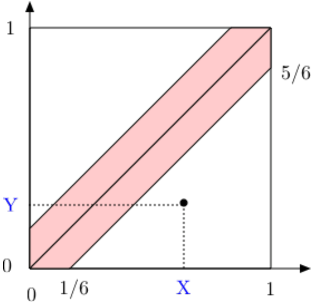

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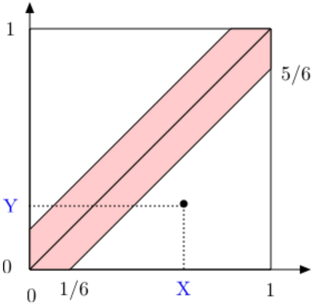

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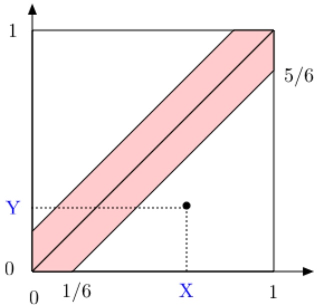

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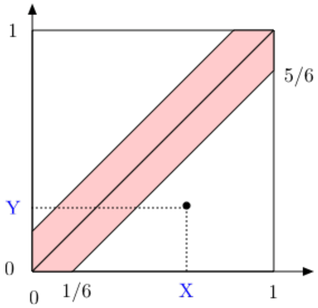

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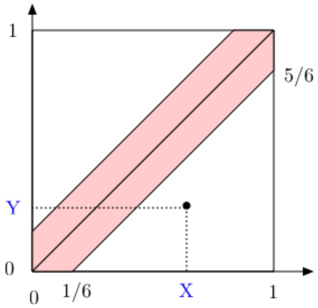

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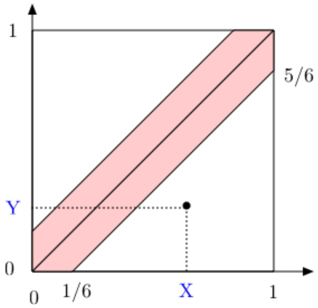

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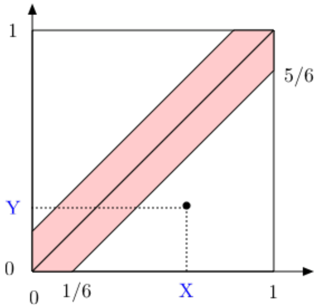

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Thus, $Pr[\text{meet}] = 1 - (\frac{5}{6})^2 = \frac{11}{36}$.

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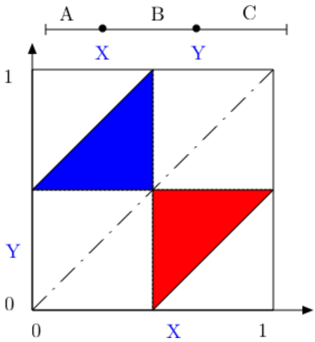

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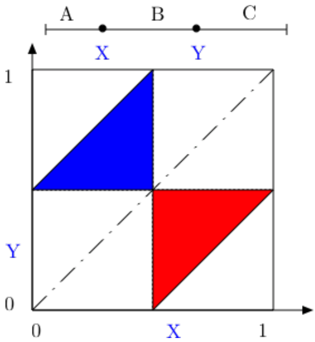

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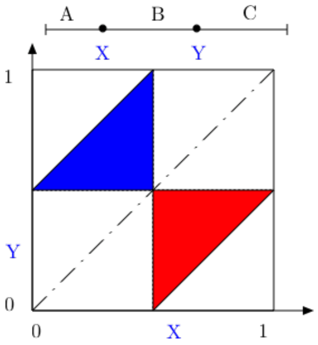

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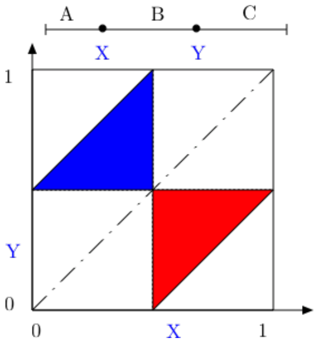

Let X, Y be the two break points along the $[0,1]$ stick.

You can make a triangle if $A < B + C, B < A + C$, and $C < A + B$.

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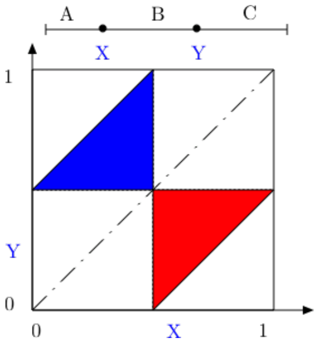

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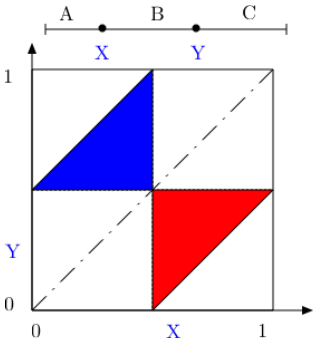

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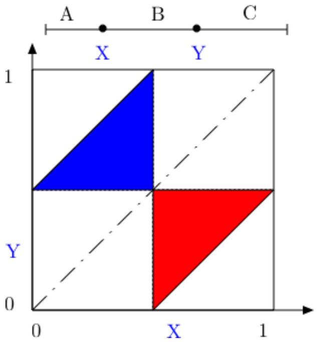

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Thus, $Pr[\text{make triangle}] = 1/4$.

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Hence,

$$E[Z] = A_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H(n).$$

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For instance, if $n = 16$, then $SNR(dB) \approx 112 dB$.

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Indeed, $(1 - \frac{a}{N})^N \approx \exp\{-a\}$.

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