CS70: Jean Walrand: Lecture 37.

Gaussian RVs and CLT

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Gaussian RVs and CLT

1. Review: Continuous Probability
2. Normal Distribution
3. Central Limit Theorem
4. Confidence Intervals
5. Bayes' Rule with Continuous RVs

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7. Variance of Sum of Independent RVs: If X_n are pairwise independent, $var[X_1 + \cdots + X_n] = var[X_1] + \cdots + var[X_n]$

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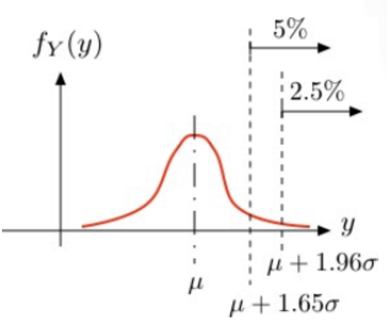

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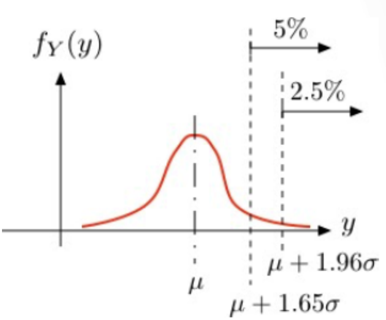

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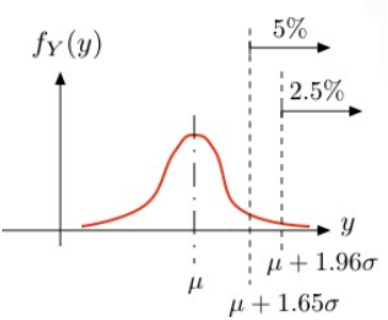

Note: $Pr[|Y - \mu| > 1.65\sigma] = 10\%$; $Pr[|Y - \mu| > 2\sigma] = 5\%$.

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Theorem If $Y = \mathcal{N}(\mu, \sigma^2)$, then

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Thus, the CLT provides a smaller confidence interval.

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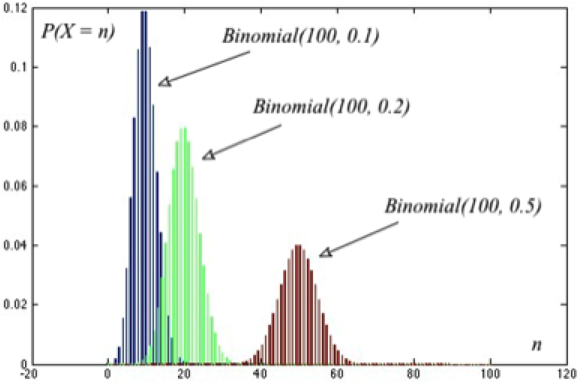

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Thus, we conclude that $Pr[\lambda^{-1} > 1] \geq 97.5\%$.

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Now,

$$\begin{aligned} E[Y|X=x] &= E[Y|A]Pr[A|X=x] + E[Y|\bar{A}]Pr[\bar{A}|X=x] \\ &= 1 \times Pr[A|X=x] + (1/3)Pr[\bar{A}|X=x] \end{aligned}$$

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Let A be the event that X, Y are $Expo(1)$.

Then,

$$\begin{aligned} Pr[A|B] &= \frac{(1/2)Pr[B|A]}{(1/2)Pr[B|A] + (1/2)Pr[B|\bar{A}]} = \frac{\exp\{-x\}\delta}{\exp\{-x\}\delta + 3\exp\{-3x\}\delta} \\ &= \frac{\exp\{-x\}}{\exp\{-x\} + 3\exp\{-3x\}} = \frac{e^{2x}}{3+e^{2x}}. \end{aligned}$$

Now,

$$\begin{aligned} E[Y|X=x] &= E[Y|A]Pr[A|X=x] + E[Y|\bar{A}]Pr[\bar{A}|X=x] \\ &= 1 \times Pr[A|X=x] + (1/3)Pr[\bar{A}|X=x] ... = \frac{1+e^{2x}}{3+e^{2x}}. \end{aligned}$$

We used $Pr[Z \in [x, x+\delta]] \approx f_Z(x)\delta$ and given A one has $f_X(x) = \exp\{-x\}$ whereas given $\bar{A}$ one has $f_X(x) = 3\exp\{-3x\}$.

Continuous RV and Bayes' Rule

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(b) $E[X] = 0.8 \times 0.5 \times \frac{2}{3} + 0.2 \times \frac{2}{3} = 0.4.$

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Gaussian and CLT

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4. Bayes' Rule: Replace $\{X = x\}$ by $\{X \in (x, x + \varepsilon)\}$.